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New Techniques in Time-Frequency Analysis

Stephen Casey
AMERICAN UNIVERSITY, THE

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New Techniques in Time-Frequency Analysis: Adaptive Band, Ultra-Wide Band and Multi-Rate Signal Processing

AFOSR Grant FA9550-12-1-0430

FINAL REPORT

Stephen D. Casey

DEPARTMENT OF MATHEMATICS
AND STATISTICS
AMERICAN UNIVERSITY
WASHINGTON, DC 20016-8050
scasey@american.edu

Abstract

The project New Techniques in Time-Frequency Analysis: Adaptive Band, Ultra-Wide Band and Multi-Rate Signal Processing led to the development of new techniques and theories in the analysis of signals. These techniques and theories were extensions of known techniques – sampling, Fourier, Gabor and wavelet analysis, and new approaches to analysis – using combinations of analysis, geometry, group theory, and number theory. **Every target item in the original statement of objectives was achieved, and several new areas of research were open up.** There were four main areas of study.

Techniques were developed to deal with classes of signals for which the known techniques have limitations, e.g., adaptive frequency band (AFB) and ultra-wide band (UWB) signals. These signals are of interest to Air Force communications systems, but the theory and techniques we have developed and propose to continue developing fit in the context of expanding the general theory. Our techniques involve signal segmentation, projection, expansion in bases, transform analysis, and multi-rate methods. Adaptive frequency band (AFB) and ultra-wide band (UWB) systems require either rapidly changing or very high sampling rates. Conventional analog-to-digital devices have non-adaptive and limited dynamic range. We investigate AFB and UWB sampling via a basis projection method. The method decomposes the signal into a basis over time segments via a continuous-time inner product operation and then samples the basis coefficients in parallel. The signal may then be reconstructed from the basis coefficients to recover the signal in the time domain. We develop the procedure of this method, analyze various methods for signal segmentation and develop a procedure to preserve orthonormality between blocks. This involves adapting an developing windowing systems for time-frequency analysis. We then demonstrate the connection of these techniques with Gabor and wavelet analysis.

We also gave a technique for multi-rate sub-Nyquist sampling. This then allows for processing of wider bandwidth signals at smaller bandwidth rates. These techniques are based on our work on multi-channel deconvolution. The "tricks" for the multi-rate theory come from clever uses of number theory and complex analysis.

We also developed a program of study connecting sampling theory with the geometry of the signal and its domain. It is relatively easy to demonstrate this connection in Euclidean spaces, but one quickly gets into open problems when the underlying space is not Euclidean. In Euclidean space, the minimal sampling rate for Paley-Wiener functions on $\mathbb{R}^d$, the Nyquist rate, is a function of the band-width. No such rate has yet been determined for hyperbolic or spherical spaces. We look to develop a structure for the tiling of frequency spaces in both Euclidean and non-Euclidean domains. In particular, we establish *Nyquist tiles* and *sampling groups* in Euclidean geometry, and discuss the extension of these concepts to hyperbolic and spherical geometry and general orientable surfaces.

We close by discussing our work in the analysis of point processes. We have developed a collection of algorithms that analyze periodic phenomena generated by either a single or multiple generators. These work even when the data is extremely sparse and noisy. The algorithms use number theory in novel ways to extract the underlying period(s) by modifying the Euclidean algorithm to determine the period from a sparse set of noisy measurements. We divide our analysis into two cases periodic processes created by a single source, and those processes created by several sources. We wish to extract the fundamental period of the generators, and, in the second case, to deinterleave the processes.

Contents

1 Original Statement of Objectives

We have investigated and propose to continue investigating new techniques in the time-frequency analysis of signals. The new techniques are developed to deal with classes of signals for which the known techniques have limitations, e.g., adaptive frequency band (AFB) and ultra-wide band (UWB) signals. These signals are of interest to Air Force communications systems, but the theory and techniques we have developed and propose to continue developing fit in the context of expanding the general theory. Our techniques involve signal segmentation, projection, expansion in bases, transform analysis and multi-rate methods.

Adaptive frequency band (AFB) and ultra-wide band (UWB) systems require either rapidly changing or very high sampling rates. Conventional analog-to-digital devices have non-adaptive and limited dynamic range. We investigate AFB and UWB sampling via a basis projection method. The method decomposes the signal into a basis over time segments via a continuous-time inner product operation and then samples the basis coefficients in parallel. The signal may then be reconstructed from the basis coefficients to recover the signal in the time domain. We develop the procedure of this method, analyze various methods for signal segmentation and develop a procedure to preserve orthonormality between blocks. We then demonstrate the connection of these techniques with Gabor and wavelet analysis. Items for the basis projection method include:

- Develop the projection method for a variety of bases, tailored to different classes of input signals. Target signals of interest to Air Force communications systems. Implement numerical models in MATLAB. Propose projection systems for chip design.
- Implement Almost Orthogonality Theory (Cotlar, Knapp, Stein) to develop B-spline windowing systems that are relatively easy to compute and almost orthogonal between blocks.

We also give a technique for multi-rate sub-Nyquist sampling. This then allows for processing of wider bandwidth signals at smaller bandwidth rates. These techniques are based on our work on multi-channel deconvolution. We give an application of these techniques to multi-rate signal streaming. In particular, we will investigate:

- Develop a systematic approach to multi-rate theory in order to develop computable systems.
- Construct multi-rate systems for signals of interest to Air Force communications. Implement numerical models in MATLAB.

We close by placing the theories in the context of standard analysis techniques. We also look into underlying principles, in particular for sampling. We propose developing a unified approach to sampling via "tiling groups." This allows us to extend sampling into non-Euclidean geometries. In particular, we outline how to develop sampling for hyperbolic and spherical geometries. Items include:

- Develop theory of "tiling groups."
- Use the theory of "tiling groups" to develop sampling formulae for hyperbolic and spherical geometries.

2 Projects Developed During the Grant

We developed new techniques and theories in the analysis of signals over the duration of the grant. These techniques and theories were extensions of known techniques – sampling, Fourier, Gabor and wavelet analysis, and new approaches to analysis – using combinations of analysis, geometry, group theory, and

number theory. **Every target item in the original statement of objectives was achieved, and several new areas of research were open up.** There were four main areas of study.

1.) Techniques were developed to deal with classes of signals for which the known techniques have limitations, e.g., adaptive frequency band (AFB) and ultra-wide band (UWB) signals. These signals are of interest to Air Force communications systems, but the theory and techniques we have developed and propose to continue developing fit in the context of expanding the general theory. Our techniques involve signal segmentation, projection, expansion in bases, transform analysis, and multi-rate methods. Adaptive frequency band (AFB) and ultra-wide band (UWB) systems require either rapidly changing or very high sampling rates. Conventional analog-to-digital devices have non-adaptive and limited dynamic range. We investigate AFB and UWB sampling via a basis projection method. We refer to the technique as *The Projection Method*. The method decomposes the signal into a basis over time segments via a continuous-time inner product operation and then samples the basis coefficients in parallel. The signal may then be reconstructed from the basis coefficients to recover the signal in the time domain. We develop the procedure of this method, analyze various methods for signal segmentation and develop a procedure to preserve orthonormality between blocks. This involves adapting an developing windowing systems for time-frequency analysis. We then demonstrate the connection of these techniques with Gabor and wavelet analysis. This work is described in Section 4.

2.) We also gave a technique for multi-rate sub-Nyquist sampling. This then allows for processing of wider bandwidth signals at smaller bandwidth rates. These techniques are based on our work on multi-channel deconvolution. The "tricks" for the multi-rate theory come from clever uses of number theory and complex analysis. This work is described in Section 5.

3.) We also developed a program of study connecting sampling theory with the geometry of the signal and its domain. It is relatively easy to demonstrate this connection in Euclidean spaces, but one quickly gets into open problems when the underlying space is not Euclidean. In Euclidean space, the minimal sampling rate for Paley-Wiener functions on $\mathbb{R}^d$, the Nyquist rate, is a function of the band-width. No such rate has yet been determined for hyperbolic or spherical spaces. We look to develop a structure for the tiling of frequency spaces in both Euclidean and non-Euclidean domains. In particular, we establish *Nyquist tiles* and *sampling groups* in Euclidean geometry, and discuss the extension of these concepts to hyperbolic and spherical geometry and general orientable surfaces. This work is described in Section 6.

4.) We close by discussing our work in the analysis of point processes. This work, although not in the original proposal, is very relevant to AFOSR interests. It was developed over the course of the grant.

We have developed a collection of algorithms that analyze periodic phenomena generated by either a single or multiple generators. These work even when the data is extremely sparse and noisy. The algorithms use number theory in novel ways to extract the underlying period(s) by modifying the Euclidean algorithm to determine the period from a sparse set of noisy measurements. We divide our analysis into two cases periodic processes created by a single source, and those processes created by several sources. We wish to extract the fundamental period of the generators, and, in the second case, to deinterleave the processes.

We first present very efficient algorithm for extracting the fundamental period from a set of sparse and noisy observations of a single source periodic process. The procedure is computationally straightforward, stable with respect to noise and converges quickly. Its use is justified by a theorem which shows that for a set of randomly chosen positive integers, the probability that they do not all share a common prime factor approaches one quickly as the cardinality of the set increases. The proof of this

theorem rests on a probabilistic interpretation of the Riemann zeta function. We then build upon this procedure to deinterleave and then analyze data from multiple periodic processes. This relies both on the probabilistic interpretation of the Riemann zeta function, the equidistribution theorem of Weyl, and Wieners periodogram. This work is described in Section 7.

3 Introduction: Background and Notation

All functions considered in this report are absolutely and square integrable functions ($f \in L^1 \cap L^2$), unless noted otherwise. References for the material on harmonic analysis include Benedetto [Ben97], Daubechies [Dau92] Dym and McKean [DM72], Grafakos [Gra14], Higgins [Hig96], Hömander [Hör90], Levin [Lev96], Papoulis [Pap62, Pap77], Young [You80] and Zayed [Zay93]. Let f be a periodic, integrable function on $\mathbb{R}$, with period 2Φ, i.e., $f \in L^1(\mathbb{T}_{2\Phi})$. The Fourier coefficients of f, $\widehat{f}[n]$, are defined by $\widehat{f}[n] = \frac{1}{2\Phi}\int_{-\Phi}^{\Phi} f(t)\exp(-i\pi nt/\Phi)\,dt$. If $\{\widehat{f}[n]\}$ is absolutely summable ($\{\widehat{f}[n]\} \in l^1$), then the Fourier series of f is $f(t) = \sum_{n\in\mathbb{Z}} \widehat{f}[n]\exp(i\pi nt/\Phi)$. Let f be a function in L^1. The Fourier transform of f is defined as $\widehat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi it\omega}dt$ for $t \in \mathbb{R}$ (time), $\omega \in \widehat{\mathbb{R}}$ (frequency). The inversion formula, for $\widehat{f} \in L^1(\widehat{\mathbb{R}})$, is $f(t) = (\widehat{f})^{\vee}(t) = \int_{\widehat{\mathbb{R}}} \widehat{f}(\omega)e^{2\pi i\omega t}d\omega$. Formally, we can think of the transform and the coefficient integral as *analysis*, and the inverse transform and series as *synthesis*. The choice to have 2π in the exponent simplifies certain expressions, e.g., for $f, g \in L^1 \cap L^2(\mathbb{R})$, $\widehat{f}, \widehat{g} \in L^1 \cap L^2(\widehat{\mathbb{R}})$, we have the *Parseval – Plancherel* equations – $\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\widehat{\mathbb{R}})}$ and $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$. We extend the transform from $L^1 \cap L^2$ to L^2 via a density argument.

Classical sampling theory applies to square integrable band-limited functions. A function that is both Ω bandlimited and L^2 has several smoothness and growth properties given in the Paley-Weiner Theorem (see, e.g., [DM72]). We define $\mathbb{PW}_\Omega = \{f : f, \widehat{f} \in L^2, \mathrm{supp}(\widehat{f}) \subset [-\Omega, \Omega]\}$. The Whittaker-Kotel'nikov-Shannon (W-K-S) Sampling Theorem applies to functions in $\mathbb{PW}_\Omega$.

Theorem 1 (W-K-S Sampling Theorem) *Let* $f \in \mathbb{PW}_\Omega$, $\mathrm{sinc}_T(t) = \frac{\sin(\frac{\pi}{T}t)}{\pi t}$ *and* $\delta_{nT}(t) = \delta(t - nT)$.

a.) *If* $T \leq 1/2\Omega$, *then for all* $t \in \mathbb{R}$,

$$f(t) = T\sum_{n\in\mathbb{Z}} f(nT)\frac{\sin(\frac{\pi}{T}(t-nT))}{\pi(t-nT)} = T\left(\left[\sum_{n\in\mathbb{Z}}\delta_{nT}\right]f\right) * \underset{T}{\mathrm{sinc}}(t).$$

b.) *If* $T \leq 1/2\Omega$ *and* $f(nT) = 0$ *for all* $n \in \mathbb{Z}$, *then* $f \equiv 0$.

4 The Projection Method

Adaptive frequency band (AFB) and ultra-wide band (UWB) systems are of interest to the ASFOSR signal processing community. They require either rapidly changing or very high sampling rates, and this severely stresses classical sampling approaches. At UWB rates, conventional analog-to-digital devices have limited dynamic range and exhibit undesired nonlinear effects such as timing jitter. Increased sampling speed leads to less accurate devices that have lower precision in numerical representation. This motivates alternative sampling schemes that use mixed-signal approaches, coupling analog processing with parallel sampling, to provide improved sampling accuracy and parallel data streams amenable to lower speed (parallel) digital computation. *Developing a sampling theory for adaptive frequency band and ultra-wide band systems is the motivation for the methods in this section of the report.* Two of the key items needed for this approach are quick and accurate computations of Fourier coefficients, which

are computed in hardware and for which we have developed a fast scheme ([Cas12, CC15, Cas15]), and effective adaptive windowing systems ([Cas12, CC15, Cas15]). Sampling schemes that work with inverse operators and multi-rate schemes were developed in [CW94, Cas98, CS99a, CW01, Cas07].

We investigate AFB and UWB sampling via a basis projection method. The method was introduced as a means of UWB parallel sampling and applied to UWB communications systems. The method first decomposes the signal into a basis over time segments via a continuous-time inner product operation and then samples the basis coefficients in parallel. The signal may then be reconstructed from the basis coefficients to recover time domain samples, or further processing may be carried out in the new domain. We address several issues associated with the basis-expansion and sampling procedure, including choice of basis and segmentation of the signal. We develop a mathematical model of the procedure, using both standard (sine, cosine) basis elements and general basis elements, and give this representation in both the time and frequency domains. First, we consider ideal rectangular windowing of each segment, and then develop a theory for overlapping windowed segments. Splines are used to analyze the degree of smoothness in time and corresponding decay in frequency that occurs with overlapping windows. We employ the theory of lapped orthogonal transforms to preserve the orthogonality of basis elements in the overlapping regions. We compare the method with traditional sampling and close by applying the procedure to binary signals. Proofs of all results and extensive error analysis of the projection method can be found in Casey and Sadler [Cas12].

The theory of splines and some techniques from ordinary differential equations give us the tools to cut up the time domain into perfectly aligned segments so that there is no loss of information. We want the systems to be smooth, so as to provide control over decay in frequency, have variable cut-off functions for flexibility in design, and adaptive, so as to adjust accordingly to changes in frequency. We also want to develop our systems so that the orthogonality of bases in adjacent and possible overlapping blocks is preserved.

Definition 1 (ON Window System) *Let* $0 < r \ll T$. *An* ON Window System *is a set of functions* $\{\mathbb{W}_k(t)\}$ *such that for all* $k \in \mathbb{Z}$

$$\begin{array}{rl} (i.) & \operatorname{supp}(\mathbb{W}_k(t)) \subseteq [kT - r, (k+1)T + r]\,, \\ (ii.) & \mathbb{W}_k(t) \equiv 1 \textit{ for } t \in [kT + r, (k+1)T - r]\,, \\ (iii.) & \mathbb{W}_k((kT + T/2) - t) = \mathbb{W}_k(t - (kT + T/2)) \textit{ for } t \in [0, T/2 + r]\,, \\ (iv.) & \sum [\mathbb{W}_k(t)]^2 \equiv 1\,, \\ (v.) & \{\widehat{[\mathbb{W}_k]^\circ}[n]\} \textit{ is absolutely summable, i.e. } \{\widehat{[\mathbb{W}_k]^\circ}[n]\} \in l^1\,. \end{array}$$

Conditions $(i.)$ and $(ii.)$ are partition properties, in that they give an exact snapshot of the input function f on $[kT + r, (k+1)T - r]$, with smooth roll-off at the edges. Condition $(iii.)$ is needed to preserve orthogonality between adjacent blocks. Condition $(iv.)$ simplifies computations, and condition $(v.)$ is needed for the computation of Fourier coefficients.

Let $\{\varphi_j(t)\}$ be an orthonormal basis for $L^2[-T/2, T/2]$. Define

$$\widetilde{\varphi_j}(t) \;=\; \begin{cases} 0 & |t| \geq T/2 + r \\ \varphi_j(t) & |t| \leq T/2 \\ -\varphi_j(-T - t) & -T/2 - r < t < -T/2 \\ \varphi_j(T - t) & T/2 < t < T/2 + r\,. \end{cases}$$

We refer to $\{\widetilde{\varphi_j}(t)\}$ as the *folded basis elements*. Let $\mathcal{T}_\alpha$ be the translation operator, i.e., $\mathcal{T}_\alpha[f](t) = f(t - \alpha)$. All of the basis elements are presented in the interval $[T/2 - r, T/2 + r]$ centered at the origin.

Therefore, the operator $\mathcal{T}_{[(k)T+T/2]}$ will place the basis in the interval $[(k)T - r, (k+1)T + r]$. In the following, $\mathbb{W}_I(t)$ is the window centered at the origin, and φ_j is a basis element in that window. Let

$$\{\Psi_{k,j}\} = \{\mathcal{T}_{[(k)T+T/2]}[\mathbb{W}_I\widetilde{\varphi_j}](t)\}\,.$$

We denote this as $\{\Psi_{k,j}\} = \{\mathbb{W}_k\widetilde{\varphi_j}(t)\}$

Theorem 2 (The Orthogonality of Overlapping Blocks) *The collection* $\{\Psi_{k,j}\} = \{\mathbb{W}_k\widetilde{\varphi_j}(t)\}$ *forms an orthonormal basis for* $L^2(\mathbb{R})$.

Given characteristics of the class of input signals, the choice of basis functions used in the projection method can be tailored to optimal representation of the signal or a desired characteristic in the signal.

Theorem 3 (Projection Formula for ON Windowing) *Let* $\{\mathbb{W}_k(t)\}$ *be an ON Window System, and let* $\{\Psi_{k,j}\}$ *be an orthonormal basis that preserves orthogonality between adjacent windows. Let* $f \in \mathbb{PW}_\Omega$ *and* $N = N(T,\Omega)$ *be such that* $\langle f, \Psi_{k,n}\rangle = 0$ *for all* $n > N$ *and all* k. *Then,* $f(t) \approx f_{\mathcal{P}}(t)$, *where*

$$f_{\mathcal{P}}(t) = \sum_{k\in\mathbb{Z}}\left[\sum_{n=-N}^{N}\langle f, \Psi_{k,n}\rangle\Psi_{k,n}(t)\right].$$

Given the flexibility of our windowing system, we can also formulate an adaptive projection system for the ON Window Systems.

Theorem 4 (Adaptive Projection Formula for ON Windowing) *Let* $f,\ \widehat{f} \in L^2(\mathbb{R})$ *and* f *have a variable but bounded band-limit* $\Omega(t)$. *Let* $\tau(t)$ *be an adaptive block of time. Let* $\{\mathbb{W}_k(t)\}$ *be an ON Window System with window size* $\tau(t) + 2r$ *on the kth block, and let* $\{\Psi_{k,j}\}$ *be an orthonormal basis that preserves orthogonality between adjacent windows. Given* $\tau(t)$, *let* $\overline{\Omega}(t) = \max\{\Omega(t) : t \in \tau(t)\}$. *Let* $N(t) = N(\tau(t), \Omega(t))$ *be such that* $\langle f, \Psi_{k,n}\rangle = 0$. *Then,* $f(t) \approx f_{\mathcal{P}}(t)$, *where*

$$f_{\mathcal{P}}(t) = \sum_{k\in\mathbb{Z}}\left[\sum_{n=-N(t)}^{N(t)}\langle f, \Psi_{k,n}\rangle\Psi_{k,n}(t)\right].$$

4.1 Remarks on Applications

Despite extensive advances in integrated circuit design and fabrication processes, wideband problems continue to hit barriers in sample and hold architectures and analog-to-digital conversion (ADC). ADC signal-to-noise and distortion ratio, the effective number of resolution bits, declines with sampling rate due to timing jitter, circuit imperfections, and electronic noise. ADC performance (speed and total integrated noise) can be improved to some extent, e.g., by cooling and therefore lowering the system temperature. However, the energy cost may be significant, and this presents a major hurdle for implementation in miniaturized devices. Digital circuitry has provided dramatically enhanced digital signal processing operation speeds, but there has not been a corresponding dramatic energy capacity increase in batteries to operate these circuits; there is no Moore's Law for batteries or ADCs.

A growing number of applications face this challenge, such as miniature and hand-held devices for communications, robotics, and micro aerial vehicles. Very wideband sensor bandwidths are desired for dynamic spectrum access and cognitive radio, radar, and ultra-wideband systems. Multi-channel and multi-sensor systems, array processing and beamforming, multi-spectral imaging, and vision systems compound the issue. All of these rely on analog sensing and a digital interface, perhaps with feedback.

This motivates mixed-signal circuit designs that tightly couple the analog and digital portions, and operate with parallel reduced bandwidth paths to relax ADC requirements. The goal of such wideband integrated circuit designs is to achieve good tradeoffs in dynamic range, bandwidth, and parallelization, while maintaining low energy consumption. This requires a careful balance of analog and digital functionality.

From a signal processing perspective, we have approached this problem by implementing an appropriate signal decomposition in the analog portion that provides parallel outputs for integrated digital conversion and processing. This naturally leads to an architecture with windowed time segmentation and parallel analog basis expansion. In this part of the report, we viewed this from the sampling theory perspective, including segmentation and window design, achieving orthogonality between segments, basis expansion and choice of basis, signal filtering, and reconstruction. The approach we have developed is tailored toward strong connections to circuit design considerations and applications. We look to generalize the windowing systems to obtain computationally efficient approaches to time-frequency analysis.

5 Multi-rate Sampling

A key step in solving the multichannel deconvolution problem involves solving an interpolation problem, reconstructing functions (the deconvolvers) in a space of restricted growth ($\widehat{\mathcal{E}'}$) from discrete data (their values on the zero sets of the convolvers). This gives solutions to the Bezout equation. This development utilizes the zero sets of the $\widehat{\mu_i}$ as different sampling rates. This connection between sampling and multichannel deconvolution naturally leads to sampling schemes on properly chosen non-commensurate lattices. Let $\{r_i\}_{i=1}^m$ be a set such that r_i/r_j is *poorly approximated by rationals* for $i \neq j$, and let $R = \sum_{i=1}^m r_i$. Let

$$\Lambda = \bigcup_{i=1}^{m} \left\{ \frac{\pm k}{2r_i} \right\}_{k\in\mathbb{N}} \bigcup \{0\}$$

be a sampling grid, made up of a union of sampling grids with non-commensurate generators $\{r_i\}$. Given a R bandlimited function f, we can reconstruct f from samples of f taken on Λ. We refer to this as the *Multi-Rate Sampling Problem* or MRSP. Our work on this can be found in [Cas16, CS00, CS99a, CS99b, CW01].

Let $t \in \mathbb{R}$, and let α be an irrational that is poorly approximated by rationals, e.g., $\sqrt{2}$, or $(1+\sqrt{5})/2$. Let $\mu_1(t) = \chi_{[-1,1]}(t)$, $\mu_2(t) = \chi_{[-\alpha,\alpha]}(t)$ model the impulse response of the channels of a two-channel system. Then

$$\widehat{\mu_1}(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta} \ , \ \widehat{\mu_2}(\zeta) = \frac{\sin(2\pi\alpha\zeta)}{\pi\zeta} \, .$$

Since α is poorly approximated by rationals, $\{\mu_i\}$ is strongly coprime. Now let

$$\Lambda_1 = \left\{ \frac{\pm n}{2} \right\} \ , \ \Lambda_2 = \left\{ \frac{\pm m}{2\alpha} \right\},$$

for $n, m \in \mathbb{N}$, and let $\Lambda = \Lambda_1 \cup \Lambda_2$. Note that the information contained in the original signal is reconstructed by creating deconvolvers defined initially on $\Lambda \cup \{0\}$. Order the elements of Λ, denoting this as $\Lambda = \{\lambda_k\}$. Walnut (see [Cas16]) has shown the following.

Theorem 5 *Let α be an irrational that is poorly approximated by rationals, and let f be a $(1+\alpha)$-band-limited function. Then f is uniquely determined by $\{f(\lambda_k)\} \cup \{f(0), f'(0)\}$.*

Note, for a $(1+\alpha)$-band-limited function, the Nyquist rate is $1/(2(1+\alpha))$. However, our individual sampling rates are $1/2$ and $1/(2\alpha)$. Both these rates are below Nyquist. The reconstruction of f from this

lattice is achieved by using complex interpolation theory. These techniques go back to basic Lagrange interpolation, and were developed for entire functions by various mathematicians, most notably B. Ya. Levin [Lev96]. Let $G(z) = \sin(2\pi z)\sin(2\pi\alpha z)$. Then, $G(z)$ is an entire function, which is almost periodic on $\mathbb{R}$, and has simple zeros on Λ, and a double zero at $\{0\}$.

Proposition 1 *Let $\lambda_k \in \Lambda$ and let*

$$H_j(z) = \frac{G(z)}{G'(\lambda_j)(z-\lambda_j)}\,.$$

Then $H_j(\lambda_k) = \delta_{jk}$.

At $z = 0$, we have to construct interpolating functions K_1, K_2 so that

$$K_1(0) = 1\,,\ K_1'(0) = 0\,,\ K_2(0) = 0\,,\ K_2'(0) = 1\,.$$

Using the Taylor expansion of G, we derive the following.

Proposition 2

$$K_1(z) = \frac{G(z)}{(G''(0)/2!)(z^2)} = \frac{G(z)}{(4\pi^2\alpha)(z^2)}\,,\ K_2(z) = \frac{G(z)}{(G''(0)/2!)(z)} = \frac{G(z)}{(4\pi^2\alpha)(z)}\,.$$

Combining these two propositions gives us the reconstruction formula.

Theorem 6 *Let α be an irrational that is poorly approximated by rationals, and let f be a $(1+\alpha)$-band-limited function. Let*

$$\Lambda_1 = \left\{\frac{\pm k}{2}\right\}\,,\ \Lambda_2 = \left\{\frac{\pm k}{2\alpha}\right\},$$

for $k \in \mathbb{N}$, and let $\Lambda = \{\lambda_k\} = \Lambda_1 \cup \Lambda_2$. Then f is uniquely determined by $\{f(\lambda_k)\} \cup \{f(0), f'(0)\}$. Moreover, f can be approximated from its values on $\Lambda \cup \{0\}$ by the formula

$$\begin{aligned} f(t) \approx & \sum_{\lambda_k \in \Lambda} f(\lambda_k)\frac{G(t)}{G'(\lambda_k)(t-\lambda_k)} \\ & + f(0)\frac{G(t)}{(4\pi^2\alpha)(t^2)} \ +\ f'(0)\frac{G(t)}{(4\pi^2\alpha)(t)}\,, \end{aligned}$$

where

$$G(t) = \sin(2\pi t)\cdot\sin(2\alpha\pi t)\,.$$

Convergence is uniform convergence on compact subsets, and is shown in a manner similar to the results for deconvolution. The proof is lengthy, and appears in [Cas16]. We need to point out two items. First, the sampling grid is rigid. Perturbation of the grid results in a loss of information. Second, that because sampling points in Λ can get arbitrarily close together ($\inf\{|\lambda_m - \lambda_n|\} = 0$, and so $\inf\{\prod_{k\neq j}|\sin(2\pi r_k\lambda)| : \lambda \in \Lambda_j\} = 0$), the set of interpolating functions can not form a Riesz basis and the interpolating formula can not converge in norm (see [Cas16]). In fact, the interpolating functions do form a Bessel sequence, but do not form a frame and therefore do not form a Riesz basis. The problems occur at points where the sampling points get close together. The interpolating function follows the original function along exactly, except for a very subtle "ripple" at those points where the sampling points get close together. The following stabilizes the construction. Let δ be given, $0 < \delta < \frac{1}{4\alpha}$. Let

$$\Lambda_\delta = \{\lambda \in \Lambda : \mathrm{dist}(\lambda, \Lambda \setminus \{\lambda\}) < \delta\}\,.$$

Elements in Λ_δ occur in pairs, with each pair containing one element from $\Lambda_1 = \left\{\frac{\pm k}{2}\right\}$ and the other from $\Lambda_2 = \left\{\frac{\pm k}{2\alpha}\right\}$ for $k \in \mathbb{N}$. Let $\{\lambda_k\} = \Lambda = \Lambda_1 \cup \Lambda_2 = \Lambda_\delta \cup {\Lambda_\delta}'$. For

$$G(t) = \sin(2\pi t) \cdot \sin(2\pi\alpha t), \text{ let } H_{\lambda_k} = \frac{G(t)}{G'(\lambda_k)(t-\lambda_k)}\,.$$

We can show that

$$\mathrm{Span}\left\{\frac{G(t)}{G'(\lambda_k)(t-\lambda_k)}\right\}$$

is dense in $L^2_{(1+\alpha)}$ and that

$$\left\{\frac{G(t)}{G'(\lambda_k)(t-\lambda_k)}\right\}_{\lambda_k \in {\Lambda_\delta}'}$$

is a Bessel sequence. Also, we can show that

$$\frac{G(t)}{4\pi\alpha t}\,,\ \frac{G(t)}{4\pi\alpha t^2} \in \mathrm{Closure}\left(\mathrm{Span}\left\{\frac{G(t)}{G'(\lambda_k)(t-\lambda_k)}\right\}\right)\,.$$

The points in Λ_δ can be treated as double points. For $\{\lambda_1, \lambda_2\} \subset \Lambda_\delta$,

$$\begin{aligned} & [f(\lambda_1)H_{\lambda_1}(t) + f(\lambda_2)H_{\lambda_2}(t)] \\ = \ & f(\lambda_1)H_{\lambda_1}(t) + f'(\xi_{\lambda_1})\left[\frac{G(t)}{\left(\frac{G'(\lambda_1)-G'(\lambda_2)}{2(\lambda_1-\lambda_2)}\right)(t-\lambda_1)(t-\lambda_2)}\right] \\ + \ & \mathcal{R}(\lambda_1-\lambda_2)^2\,, \end{aligned}$$

where $\xi_{\lambda_1} \longrightarrow \lambda_1$ and $\mathcal{R}(\lambda_1-\lambda_2)^2 \longrightarrow 0$ as $\delta \longrightarrow 0$. Finally, we can show that

$$\left\{\frac{G(t)}{(G''(\lambda_k)/2!)(t-\lambda_k)^2}\right\}_{\lambda_k \in \Lambda_\delta}$$

is a Bessel sequence.

The result generalizes. We can create sampling sets on ℓ lattices using a set of numbers $\{r_i\}_{i=1}^{\ell}$ such that (r_i/r_j) is poorly approximated by rationals for $i \neq j$. Again, if $\{p_i\}_{i=1}^{\ell-1}$ is a set of primes,

$$\{1, \sqrt{p_1}, \sqrt{p_1 p_2}, \ldots, \sqrt{p_1 p_2 \cdots p_{\ell-1}}\}$$

is a set of numbers whose ratios are poorly approximated by rationals. Let $\Lambda_k = \left\{\frac{\pm n}{2r_k}\right\}$ for $n \in \mathbb{N}$, and let

$$\bigcup_{k=1}^{\ell} \Lambda_k = \Lambda = \{\lambda_k\}\,.$$

We reconstruct on $\Lambda \cup \{0\}$, letting

$$G(z) = \prod_{k=1}^{\ell} \sin(2\pi r_k z)$$

and letting

$$H_m(z) = \frac{G(z)}{G'(\lambda_m)(z-\lambda_m)}\,.$$

Then $H_m(\lambda_n) = \delta_{mn}$. The interpolating functions at the origin are a linear combination of

$$g_j = \frac{G(z)}{(z^j)}\,,\ j = 1, \ldots, \ell\,,$$

chosen so that

$$K_k^{(j-1)}(0) = \delta_{kj}\,,\ k, j = 1, \ldots, \ell\,.$$

As before, using Taylor series, we get

$$\begin{aligned} H_k^{(j)}(t) &= 2^n \pi^n \frac{(n-k)!}{(n-k-j)!} \sqrt{p_1 p_2 \cdots p_n} t^{n-k-j} \\ &\quad - \frac{2^{n+2}\pi^{n+2}}{3!} \frac{(n-k+2)!}{(n-k-j+2)!} \sqrt{p_1 p_2 \cdots p_n} \\ &\quad (1 + \sum_{i=1}^{n-1} p_i) t^{n-k-j+2} \\ &\quad + O(t^{n-k-j+4})\,. \end{aligned}$$

We want

$$K_k(t) = \sum_{i=1}^{k} c_i H_i(t) \text{ so that } K_k^{(j-1)}(0) = \delta_{kj}\,,\ k, j = 1, \ldots, \ell\,.$$

Solving these relationships gives

$$\begin{aligned} K_k(t) &= \sum_{i=1}^{k} c_i H_i(t) \\ &= \frac{\sum_{m=0}^{\lceil (n-k)/2 \rceil} (\frac{2^2\pi^2}{3!}(1 + \sum_{i=1}^{n-1} p_i))^{2m} t^{2m} G(t)}{\frac{G^{(n)}(0)}{(n)!} \cdot (k-1)! \cdot t^{n-k+1}}\,. \end{aligned}$$

6 Sampling and Non-Euclidean Geometry

The study of non-Euclidean geometries is becoming increasingly relevant in signal processing. There are numerous motivations for extending signal processing, and in particular, sampling theory, to non-Euclidean spaces, and in particular, hyperbolic and spherical geometries. Hyperbolic space and its importance in Electrical Impedance Tomography (EIT) [BCT96, Ber01] and Network Tomography [BGB06] has been mentioned in several papers of Berenstein et. al. and some methods developed in papers of Kuchment, e.g., [Kuc06]. Irregular sampling of band-limited functions by iteration in hyperbolic space is possible, as shown by Feichtinger and Pesenson [FP04, FP05] and Christensen and Ólafsson [CÓ13]. Applications where data are defined inherently on the sphere are found in computer graphics, planetary science, geophysics, quantum chemistry, and astrophysics [DH94, MW11]. In many of these applications, a harmonic analysis of the data is insightful. For example, spherical harmonic analysis has been remarkably successful in cosmology, leading to the emergence of a standard cosmological model [CW98, MW11].

Our approach is to connect sampling theory with the geometry of the signal and its domain [CC16]. It is relatively easy to demonstrate this connection in Euclidean spaces, but one quickly gets into open problems when the underlying space is not Euclidean. For Paley-Wiener functions on $\mathbb{R}$ the minimal sampling rate, the *Nyquist Rate*, is a function of the band-width. The establishment of the exact Nyquist rate in non-Euclidean spaces is an open problem. This rate is needed to develop regular sampling. We

use two tools to work on the problem – the Beurling-Landau density and Voronoi cells. Using these tools, we establish a relation in Euclidean domains, connecting Beurling-Landau density to sampling lattices and hence dual lattice groups, and then use these dual lattices to define Voronoi cells, which become our tiles in frequency. We then discuss how to extend this to non-Euclidean geometries.

Classical sampling theory applies to square integrable band-limited functions. A function that is both Ω bandlimited and L^2 has several smoothness and growth properties given in the Paley-Weiner Theorem (see, e.g., [DM72]). We define $\mathbb{PW}_\Omega = \{f : f, \widehat{f} \in L^2, \mathrm{supp}(\widehat{f}) \subset [-\Omega, \Omega]\}$. The Whittaker-Kotel'nikov-Shannon (W-K-S) Sampling Theorem applies to functions in $\mathbb{PW}_\Omega$.

Theorem 7 (W-K-S Sampling Theorem) *Let* $f \in \mathbb{PW}_\Omega$, $\mathrm{sinc}_T(t) = \frac{\sin(\frac{\pi}{T}t)}{\pi t}$ *and* $\delta_{nT}(t) = \delta(t - nT)$.

a.) *If* $T \leq 1/2\Omega$, *then for all* $t \in \mathbb{R}$,

$$f(t) = T \sum_{n\in\mathbb{Z}} f(nT) \frac{\sin(\frac{\pi}{T}(t - nT))}{\pi(t - nT)} = T \left(\left[\sum_{n\in\mathbb{Z}} \delta_{nT} \right] f \right) * \underset{T}{\mathrm{sinc}}(t)\,.$$

b.) *If* $T \leq 1/2\Omega$ *and* $f(nT) = 0$ *for all* $n \in \mathbb{Z}$, *then* $f \equiv 0$.

The sphere $\mathbb{S}^2$ is compact, and its study requires different tools. Fourier analysis on $\mathbb{S}^2$ amounts to the decomposition of $L^2(\mathbb{S}^2)$ into minimal subspaces invariant under all rotations in $SO(3)$. Band-limited functions on the sphere are spherical polynomials. Sampling on the sphere is how to sample a band-limited function, an Nth degree spherical polynomial, at a finite number of locations, such that all of the information content of the continuous function is captured. Since the frequency domain of a function on the sphere is discrete, the spherical harmonic coefficients describe the continuous function exactly. A sampling theorem thus describes how to exactly recover the spherical harmonic coefficients of the continuous function from its samples. Developing sampling lattices leads to questions on how to efficiently tile the sphere, a subject in its own right, e.g., Driscoll and Healy [DH94], Keiner, Kunis, and Potts [KKP07], and McEwen and Wiaux [MW11].

The Nyquist rate allows us to develop an efficient tiling of frequency space. A *tiling* or a tessellation of a flat surface is the covering of the plane or region in the plane using one or more geometric shapes, called tiles, with no overlaps and no gaps. This generalizes to higher dimensions. We look to develop *Nyquist tiles* and *sampling groups* for Euclidean, hyperbolic, and spherical spaces and general analytic surfaces. We first assume that signals are single band and symmetric in frequency, i.e., that the transform of the signal can be contained in a simply connected region centered at the origin. Symmetry can be achieved by shifting, and multi-band signals can be addressed by the techniques in this report.

Consider sampling in $\mathbb{R}$. Given $f \in \mathbb{PW}_\Omega$ choose $T \leq 1/(2\Omega)$. For $f \in L^1([0, T))$ the *T-periodization* of f is $(f_T)^\circ(t) = \sum_{n\in\mathbb{Z}} f(t - nT)$. We can then expand $(f_T)^\circ(t)$ in a Fourier series. The sequence of Fourier coefficients of this T-periodic function are given by $\widehat{(f_T)^\circ}[n] = \frac{1}{T}\widehat{f}\left(-\frac{n}{T}\right)$. Then if the value of the Fourier series of $f_T{}^\circ$ agrees with the value of $f_T{}^\circ$ at the origin, we get $T\sum_{n\in\mathbb{Z}} f(nT) = \sum_{n\in\mathbb{Z}} \widehat{f}(n/T)$PSF1 . Thus, the Poisson Summation Formula allows us to compute the Fourier series of $(f_T)^\circ$ in terms of the Fourier transform of f at equally spaced points. This extends to the Schwartz class of distributions as $\widehat{\sum_{n\in\mathbb{Z}} \delta_{nT}} = \frac{1}{T}\sum_{n\in\mathbb{Z}} \delta_{n/T}$PSF2 . Then, for $f \in \mathbb{PW}_\Omega$, if we sample at exactly Nyquist

$$f(t) = \frac{1}{2\Omega} \left(\left[\sum_{n\in\mathbb{Z}} \delta_{\left(\frac{1}{2\Omega}\right)} \right] f \right) * \underset{\left(\frac{1}{2\Omega}\right)}{\mathrm{sinc}}(t)$$

if and only if

$$\widehat{f}(\omega) = \left(\sum_{n\in\mathbb{Z}} \widehat{f}(\omega - 2n\Omega)\right) \cdot \chi_{[-\Omega,\Omega)}(\omega) = \left(\sum_{n\in\mathbb{Z}} \left[\delta_{2n\Omega}\right] \widehat{f}\right) \cdot \chi_{[-\Omega,\Omega)}(\omega) .$$

The interval $[-\Omega, \Omega)$ is simply connected and symmetric to the origin. It is spread by the group of translations to form a tiling of frequency space – $\{[(k-1)\Omega, (k+1)\Omega)\}$. We refer to $[-\Omega, \Omega)$ as a *sampling interval*. Note, sampling intervals are "half open, half closed," with length determined by the Nyquist rate. The inverse transform of the characteristic functions of the tiles are sinc functions, which form an orthonormal (o.n.) basis for $\mathbb{PW}_\Omega$. Sampling is expressed in terms of this basis. We can now define the following.

Definition 2 (Nyquist Tiles for $f \in \mathbb{PW}_\Omega$) *Let f be a non-trivial function in $\mathbb{PW}_\Omega$. The* Nyquist Tile $\mathbb{NT}(f)$ *for f is the sampling interval of minimal length in $\widehat{\mathbb{R}}$ such that* $\operatorname{supp}(\widehat{f}) \subseteq \mathbb{NT}(f)$. *A* Nyquist Tiling *for f is the set of translates $\{\mathbb{NT}(f)_k\}_{k\in\mathbb{Z}}$ of Nyquist tiles which tile $\widehat{\mathbb{R}}$.*

The Nyquist tile is transported by a group of motions to cover the transform domain.

Definition 3 (Sampling Group for $f \in \mathbb{PW}_\Omega$) *Let $f \in \mathbb{PW}_\Omega$ with Nyquist Tile $\mathbb{NT}(f)$. The* Sampling Group $\mathbb{G}(f)$ *is a group of translations such that $\mathbb{NT}(f)$ tiles $\widehat{\mathbb{R}}$.*

We now extend these ideas to k-dimensional Euclidean space. Let $T > 0$ and let $f(t)$ be a function such that $\operatorname{supp} f \subseteq [0, T]^k$. The T-*periodization* of f is $[f]^\circ(t) = \sum_{n\in\mathbb{Z}^d} f(t - nT)$. We can expand a T-periodic function $[f]^\circ(t)$ in a Fourier series. Denote the lattice $\Lambda = \mathbf{T}\mathbb{Z}^d$, where $\mathbf{T}$ is the $n \times n$ matrix with T on the main diagonal and zeroes elsewhere. The sequence of Fourier coefficients of this periodic function on the lattice $\Lambda = \mathbf{T}\mathbb{Z}^d$ are given by $\widehat{[f]^\circ}[n] = \frac{1}{T^d}\widehat{f}\left(-\frac{n}{T}\right)$. We have $\sum_{n\in\mathbb{Z}^d} f(t + nT) = \frac{1}{T^d}\sum_{n\in\mathbb{Z}^d} \widehat{f}(n/T)e^{2\pi i n\cdot t/T}$. Therefore, $\sum_{n\in\mathbb{Z}^d} f(nT) = \frac{1}{T^d}\sum_{n\in\mathbb{Z}^d} \widehat{f}(n/T)\text{PSF1}$. We can write the Poisson summation formula for an arbitrary lattice by a change of coordinates. Let $\mathbf{A}$ be an invertible $d\times d$ matrix, $\Lambda = \mathbf{A}\,\mathbb{Z}^d$, and $\Lambda^\perp = (\mathbf{A}^T)^{-1}\mathbb{Z}^d$ be the dual lattice. Then

$$\sum_{\lambda\in\Lambda} f(t+\lambda) = \sum_{n\in\mathbb{Z}^d} (f \circ \mathbf{A})(\mathbf{A}^{-1}t + n) = \frac{1}{|\det \mathbf{A}|}\sum_{n\in\mathbb{Z}^d} \widehat{f}((\mathbf{A}^T)^{-1}(n))e^{2\pi i(\mathbf{A}^T)^{-1}(n)\cdot t} .$$

Note, $|\det \mathbf{A}| = \operatorname{vol}(\Lambda)$. This last expression can be expressed more directly as $\sum_{\lambda\in\Lambda} f(t+\lambda) = \frac{1}{\operatorname{vol}(\Lambda)}\sum_{\beta\in\Lambda^\perp} \widehat{f}(\beta)e^{2\pi i\beta\cdot t}$. This extends again to the Schwartz class of distributions as $\widehat{\sum_{\lambda\in\Lambda}\delta_\lambda} = \frac{1}{\operatorname{vol}(\Lambda)}\sum_{\beta\in\Lambda^\perp}\delta_\beta\,\text{PSF2}$.

The sampling formula again follows from computations and an application of (PSF2). We assume a single band signal. Let Λ be a regular sampling lattice in $\mathbb{R}^d$,and let $\Lambda^\perp$ be the dual lattice in $\widehat{\mathbb{R}^d}$. Then Λ has generating vectors $\{\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, \ldots, \boldsymbol{\tau_d}\}$, and the sampling lattice can be written as $\Lambda = \{\lambda : \lambda = z_1\boldsymbol{\tau_1} + z_2\boldsymbol{\tau_2} + \ldots + z_d\boldsymbol{\tau_d}\}$ for $(z_1, z_2, \ldots, z_d) \in \mathbb{Z}^d$. Let $\{\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \ldots, \boldsymbol{\Omega}_d\}$ be the generating vectors for the dual lattice $\Lambda^\perp$. The dual sampling lattice can be written as $\Lambda^\perp = \{\lambda^\perp : \lambda^\perp = z_1\boldsymbol{\Omega}_1 + z_2\boldsymbol{\Omega}_2 + \ldots + z_d\boldsymbol{\Omega}_d\}$ for $(z_1, z_2, \ldots, z_d) \in \mathbb{Z}^d$. The vectors $\{\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \ldots, \boldsymbol{\Omega}_d\}$ generate a parallelepiped. We want to use this parallelepiped to create a tiling, and therefore we make the parallelepiped "half open, half closed" as follows. If we shift the parallelepiped so that one vertex is at the origin, we include all of the boundaries that contain the origin, and exclude the other boundaries. We denote this region as a *sampling parallelepiped* $\boldsymbol{\Omega}_{\mathcal{P}}$. If the region $\boldsymbol{\Omega}_{\mathcal{P}}$ is a hyper-rectangle, we get the familiar sampling formula

$$f(t) = \frac{1}{\operatorname{vol}(\Lambda)}\sum_{n\in\mathbb{Z}^d} f(\frac{n_1}{\omega_1}, \ldots, \frac{n_d}{\omega_d})\frac{\sin(\frac{\pi}{\omega_1}(t - n_1\omega_1))}{\pi(t - n_1\omega_1)} \cdot \ldots \cdot \frac{\sin(\frac{\pi}{\omega_d}(t - n_d\omega_d))}{\pi(t - n_d\omega_d)} .$$

If, however, the sampling parallelepiped $\mathbf{\Omega}_{\mathcal{P}}$ a general parallelepiped, we first have to compute the inverse Fourier transform of $\chi_{\mathbf{\Omega}_{\mathcal{P}}}$. Let $\mathcal{S}$ be the generalized sinc function $\mathcal{S} = \frac{1}{\mathrm{vol}(\Lambda)}(\chi_{\mathbf{\Omega}_{\mathcal{P}}})^{\vee}$. Then, the sampling formula (see [Hig96]) becomes $f(t) = \sum_{\lambda\in\Lambda} f(\lambda)\mathcal{S}(t-\lambda)$.

Definition 4 (Nyquist Tiles for $f \in \mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$) *Let*

$$\mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}} = \{f \text{ continuous} : f \in L^2(\mathbb{R}^d), \widehat{f} \in L^2(\widehat{\mathbb{R}^d}), \mathrm{supp}(\widehat{f}) \subset \mathbf{\Omega}_{\mathcal{P}}\},$$

where $\{\mathbf{\Omega}_1, \mathbf{\Omega}_2, \ldots, \mathbf{\Omega}_d\}$ *be the generating vectors for the dual lattice* $\Lambda^{\perp}$*. Let* f *be a non-trivial function in* $\mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$*. The* Nyquist Tile $\mathbb{NT}(f)$ *for* f *is the sampling parallelepiped of minimal area in* $\widehat{\mathbb{R}^d}$ *centered at the origin such that* $\mathrm{supp}(\widehat{f}) \subseteq \mathbb{NT}(f)$*. A* Nyquist Tiling *is the set of translates* $\{\mathbb{NT}(f)_k\}_{k\in\mathbb{Z}^d}$ *of Nyquist tiles which tile* $\widehat{\mathbb{R}^d}$*.*

Definition 5 (Sampling Group for $f \in \mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$) *Let* $f \in \mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$ *with* Nyquist Tile $\mathbb{NT}(f)$*. The* Sampling Group $\mathbb{G}$ *is a symmetry group of translations such that* $\mathbb{NT}(f)$ *tiles* $\widehat{\mathbb{R}^d}$*.*

A sequence Λ is *separated* or *uniformly discrete* if $q = \inf_k(\lambda_{k+l} - \lambda_k) > 0$. The value q is referred to as the *separation constant* of Λ. With a separated sequence Λ we associate a distribution function $n_\Lambda(t)$ defined such that for $a < b$, $n_\Lambda(b) - n_\Lambda(a) = \mathrm{card}(\Lambda \cap (a,b])$, and normalized such that $n_\Lambda(0) = 0$. There is clearly a one-to-one correspondence between Λ and n_Λ. A discrete set Λ is a *set of sampling* for $\mathbb{PW}_\Omega$ if there exists a constant C such that $\|f\|_2^2 \leq C\sum_{\lambda_k\in\Lambda} |f(\lambda_k)|^2$ for every $f \in \mathbb{PW}_\Omega$. The set Λ is called a *set of interpolation* for $\mathbb{PW}_\Omega$ if for every square summable sequence $\{a_\lambda\}_{\lambda\in\Lambda}$, there is a solution $f \in \mathbb{PW}_\Omega$ to $f(\lambda) = a_\lambda$, $\lambda \in \Lambda$. Clearly, all complete interpolating sequences are separated. Landau showed that if Λ is a sampling sequence for $\mathbb{PW}_\Omega$, then there exists constants A and B, independent of a, b such that $n_\Lambda(b) - n_\Lambda(a) \geq (b-a) - A\log^+(b-a) - B$.

Definition 6 (Beurling-Landau Densities) *The* Beurling-Landau lower D^- and upper D^+ densities *are given by*

$$D^-(\Lambda) = \liminf_{r\to\infty} \inf_{t\in\mathbb{R}} \frac{(n_\Lambda(t+r)) - n_\Lambda(t)}{r}, \; D^+(\Lambda) = \limsup_{r\to\infty} \sup_{t\in\mathbb{R}} \frac{(n_\Lambda(t+r)) - n_\Lambda(t)}{r}$$

The densities are defined similarly in higher dimensions. Specifically, for the exact and stable reconstruction of a band-limited function f from its samples $\{f(\lambda_k) : \lambda_k \in \Lambda\}$, it is sufficient that the Beurling-Landau lower density satisfies $D^-(\Lambda) > 1$. A set fails to be a sampling set if $D^-(\Lambda) < 1$. Conversely, if f is uniquely and stably determined by its samples on Λ, then $D^-(\Lambda) \geq 1$. Note, a sampling set for which the reconstruction is stable in this sense is called a (stable) set of sampling. This terminology is used to contrast a set of sampling with the weaker notion of a set of uniqueness. Λ is a set of uniqueness for $\mathbb{PW}_\Omega$ if $f|_\Lambda = 0$ implies that $f = 0$. Whereas a set of sampling for $\mathbb{PW}_\Omega$ has a density $D^-(\Lambda) \geq 1$, there are sets of uniqueness with arbitrarily small density. We also have that if the Beurling-Landau upper density satisfies $D^+(\Lambda) \leq 1$, then Λ is a set of interpolation.

The canonical case is when $\Omega = 2\pi$ and $\Lambda = \mathbb{Z}$. Since $\{e^{int}\}$ in an o.n. basis for $L^2[-\pi,\pi]$, it follows from Parseval that Λ is both a set of sampling and a set of interpolation. This scales by a change of variable, and so $\Lambda = \frac{1}{\Omega}\mathbb{Z}$ is both a set of sampling and a set of interpolation for $\mathbb{PW}_{2\pi\Omega}$. Moreover, general lattices can be compared to the canonical results as follows. If Λ is a set of sampling for $\mathbb{PW}_{2\pi\Omega}$, then Λ is everywhere at least as dense as the lattice $\frac{1}{\Omega}\mathbb{Z}$. If Λ is a set of interpolation for $\mathbb{PW}_{2\pi\Omega}$, then Λ is everywhere at least as sparse as the lattice $\frac{1}{\Omega}\mathbb{Z}$.

This generalizes to $\mathbb{R}^d$. Let $\mathbf{\Omega}_{\mathcal{P}}$ be a hyper-rectangle with side lengths Ω. If we normalize the density of $\mathbb{Z}^d$ to be one, then the density of the canonical lattice for $\mathbb{PW}_{2\pi\mathbf{\Omega}_{\mathcal{P}}}$ is $1/(2\pi)^d$ times the volume of the

spectrum $\mathbf{\Omega}_{\mathcal{P}}$. Then, if Λ is a set of sampling for $\mathbb{PW}_{2\pi\mathbf{\Omega}_{\mathcal{P}}}$, then Λ is everywhere at least as dense as the lattice $\frac{1}{\Omega^d}\mathbb{Z}^d$. If Λ is a set of interpolation for $\mathbb{PW}_{2\pi\mathbf{\Omega}_{\mathcal{P}}}$, then Λ is everywhere at least as sparse as the lattice $\frac{1}{\Omega^d}\mathbb{Z}^d$.

6.1 Voronoi Cells for Euclidean Space

We use our sampling lattices to develop *Voronoi cells* corresponding to the sampling lattice. These cells will be, in the Euclidean case, our Nyquist tiles.

Definition 7 (Voronoi Cells in $\widehat{\mathbb{R}^d}$) *Let $\widehat{\Lambda} = \{\widehat{\lambda}_k \in \widehat{\mathbb{R}^d} : k \in \mathbb{N}\}$ be a discrete set in $\widehat{\mathbb{R}^d}$. Then, the Voronoi cells $\{\Phi_k\}$, the Voronoi partition $\mathcal{VP}(\widehat{\Lambda})$, and partition norm $\|\mathcal{VP}(\widehat{\Lambda})\|$ corresponding to this set are defined as follows. Here,* dist *is the Euclidean distance.*

1.) *The* Voronoi cells $\Phi_k = \{\omega \in \widehat{\mathbb{R}^d} : \mathrm{dist}(\omega, \widehat{\lambda}_k) \leq \inf_{j\neq k} \mathrm{dist}(\omega, \widehat{\lambda}_j)\}$,

2.) *The* Voronoi partition $\mathcal{VP}(\widehat{\Lambda}) = \{\Phi_k \in \widehat{\mathbb{R}^d}\}_{k\in\mathbb{Z}^d}$,

3.) *The* partition norm $\|\mathcal{VP}(\widehat{\Lambda})\| = \sup_{k\in\mathbb{Z}^d}\sup_{\omega,\nu\in\Phi_k} \mathrm{dist}(\omega,\nu)$.

Given $f, \widehat{f} \in L^2(\mathbb{R}^d)$ such that $f \in \mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$, if the signal is sampled on a lattice exactly at Nyquist, we get a sampling grid $\Lambda = \{\lambda_k \in \mathbb{R}^d\}_{k\in\mathbb{Z}^d}$ that is both a sampling set and a set of interpolation. The Beurling-Landau lower density and the Beurling-Landau upper density are equal for Λ. The dual lattice $\Lambda^\perp$ in frequency space can be used to create Voronoi cells $\{\Phi_k\}$, a Voronoi partition $\mathcal{VP}(\Lambda^\perp)$, and partition norm $\|\mathcal{VP}(\Lambda^\perp)\|$. If we sample on a lattice exactly at Nyquist, each sample point will correspond to an element in the dual lattice which is at the center of a Nyquist tile $\mathbb{NT}(f)$ for f. The set of Nyquist tiles will cover $\widehat{\mathbb{R}^d}$. If, however, we develop the Voronoi cells $\{\Phi_k\}$ for $\Lambda^\perp$, we get $\mathcal{VP}(\Lambda^\perp) = \{\Phi_k \in \widehat{\mathbb{R}^d}\}_{k\in\mathbb{Z}^d}$ such that for all k, $\Phi_k = \{\omega \in \widehat{\mathbb{R}^d} : \mathrm{dist}(\omega, \lambda_k^\perp) \leq \inf_{j\neq k} \mathrm{dist}(\omega, \lambda_j^\perp)\}$. But this puts $\lambda_k^\perp$ in the center of the cell. Then, if we construct the Voronoi cell containing this point, we will get, up to the boundary, the exact Nyquist tile corresponding to this point. Nyquist tiles are "half open, half closed." If we shift a Nyquist tile so that one vertex is at the origin, we include all of the boundaries that contain the origin, and exclude the other boundaries. To get the exact correspondence between $\mathbb{NT}(f)_k$ and Φ_k, we make Φ_k "half open, half closed," and denote it as $\widetilde{\Phi_k}$. We denote the adjusted Voronoi partition as $\widetilde{\mathcal{VP}}$.

Theorem 8 (Nyquist Tiling for Euclidean Space) *Let f be a non-trivial function in $\mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$, and let $\Lambda = \{\lambda_k \in \mathbb{R}^d\}_{k\in\mathbb{Z}^d}$ be the sampling grid which samples f exactly at Nyquist. Let $\Lambda^\perp$ be the dual lattice in frequency space. Then the adjusted Voronoi partition $\widetilde{\mathcal{VP}}(\Lambda^\perp) = \{\widetilde{\Phi_k} \in \widehat{\mathbb{R}^d}\}_{k\in\mathbb{Z}^d}$ equals the Nyquist Tiling, i.e.,*

$$\{\widetilde{\Phi_k} \in \widehat{\mathbb{R}^d}\}_{k\in\mathbb{Z}^d} = \{\mathbb{NT}(f)_k\}_{k\in\mathbb{Z}^d}\,.$$

Moreover, the partition norm equals the volume of $\Lambda^\perp$, i.e.,

$$\|\widetilde{\mathcal{VP}}(\Lambda^\perp)\| = \sup_{k\in\mathbb{Z}^d}\sup_{\omega,\nu\in\widetilde{\Phi_k}} \mathrm{dist}(\omega,\nu) = \mathrm{vol}(\Lambda^\perp)\,,$$

and the sampling group $\mathbb{G}$ is exactly the group of motions that preserve $\Lambda^\perp$.

This connects, in the Euclidean case, sampling theory with the geometry of the signal and its domain. Given a function $f \in \mathbb{PW}_\Omega$, sampling of such a function is the process of tiling the frequency domain by translated identical copies of the parallelepiped of minimal area, the *Nyquist Tile*, which contains

the frequency support of $\widehat{f}$. The relation between the geometry and sampling problem in the Euclidean case is as follows: the set of the corresponding translations – the *Sampling Group* – forms a symmetry group. The corresponding sampling set, which is simply the annihilator of the sampling group, is also a symmetry group of translations on $\mathbb{R}^d$. The set of copies of the Nyquist tile, obtained by applying the sampling group, is the *Nyquist Tiling*.

The situation is considerably different when the underlying space is not Euclidean. We quickly get into open problems. Theorem 8 gives an approach for solving the problem in non-Euclidean spaces. We suggest using the two tools we just established – the Beurling-Landau density and Voronoi cells. Our next section discusses the geometry of orientable surfaces. In particular, it provides insight into why a focus on Euclidean, spherical, and especially, hyperbolic geometries is important.

6.2 Sampling in Hyperbolic Space

We begin by stating the Fourier transform, its inversion, and the Plancherel formula for hyperbolic space [Hel00].

Let dz denote the area measure on the unit disc $\mathbb{D} = \{z \mid |z| < 1$, and let the measure dv be given by then the SU$(1,1)$-invariant measure on $\mathbb{D}$ is given by $dv(z) = dz/(1-|z|^2)^2$. For functions $f \in L^1(\mathbb{D}, dv)$ the *Helgason-Fourier transform* is defined as

$$\widehat{f}(\lambda, b) = \int_{\mathbb{D}} f(z) e^{(-i\lambda+1)\langle z,b\rangle}\, dv(z)$$

for $\lambda > 0$ and $b \in \mathbb{T}$. Here $\langle z, b\rangle$ denotes the minimal hyperbolic distance from the origin to the horocycle through z and a point $b \in \partial\mathbb{D}$. The mapping $f \mapsto \widehat{f}$ extends to an isometry $L^2(\mathbb{D}, dv) \to L^2(\mathbb{R}^+ \times \mathbb{T}, (2\pi)^{-1}\lambda \tanh(\lambda\pi/2) d\lambda\, db)$, i.e., the Plancherel formula becomes

$$\int_{\mathbb{D}} |f(z)|^2 \frac{dz}{(1-|z|^2)^2} = \frac{1}{2\pi} \int_{\mathbb{R}^+\times\mathbb{T}} |\widehat{f}(\lambda, b)|^2 \lambda \tanh(\lambda\pi/2) d\lambda\, db.$$

Here db denotes the normalized measure on the circle $\mathbb{T}$, such that $\int_{\mathbb{T}} db = 1$, and $d\lambda$ is Lebesgue measure on $\mathbb{R}$. The *Helgason-Fourier inversion formula* is

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{R}^+} \int_{\mathbb{T}} \widehat{f}(\lambda, b) e^{(i\lambda+1)\langle z,b\rangle} \lambda \tanh(\lambda\pi/2)\, d\lambda\, db\,.$$

A function $f \in L^2(\mathbb{D}, dv)$ is called *band-limited* if its Helgason-Fourier transform $\widehat{f}$ is supported inside a bounded subset $[0, \Omega]$ of $\mathbb{R}^+$. The collection of band-limited functions with band-limit inside a set $[0, \Omega]$ will be denoted $\mathbb{PW}_\Omega = \mathbb{PW}_\Omega(\mathbb{D})$. This definition of band-limit coincides with the definitions given in [FP04] and [CÓ13] which both show that sampling is possible for band-limited functions. The Laplacian on $\mathbb{D}$ is symmetric and given by

$$\Delta = (1-x^2-y^2)^{-2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

and we note that

$$\widehat{\Delta f}(\lambda, b) = -(\lambda^2+1)\widehat{f}(\lambda, b).$$

Therefore, if $f \in \mathbb{PW}_\Omega(\mathbb{D})$, we see that the following Bernstein inequality is satisfied $\|\Delta^n f\| \leq (1 + |\Omega|^2)^{n/2} \|f\|$.

In the following section we will describe sampling results for band-limited functions on hyperbolic space, which, it must be stressed, do not deal with optimal densities.

6.2.1 Sampling via Operator Theory in $\mathbb{D}$

The work in [FP04] defines band-limits using the spectrum of the Laplacian on a manifold, while [CÓ13] builds on representation theory which for the case at hand gives the explicit form of the Fourier transform on $\mathbb{D}$ as defined above. We also refer to the paper [FP05] which provides the same results in the setting of the upper half plane (which is bi-holomorphically equivalent to $\mathbb{D}$). These papers build on Neumann series for an operator based on sampling as well as the Bernstein inequality. The sampling operators have previously been explored in [Grö91, Grö92].

According to Pesenson [Pes00] there is a natural number N such that for any sufficiently small r there are points $x_j \in \mathbb{D}$ for which $B(x_j, r/4)$ are disjoint, $B(x_j, r/2)$ cover $\mathbb{D}$ and $1 \leq \sum_j \chi_{B(x_j,r)} \leq N$. Such a collection of $\{x_j\}$ will be called an (r, N)-lattice.

Let ϕ_j be smooth non-negative functions which are supported in $B(x_j, r/2)$ and satisfy that $\sum_j \phi_j = 1_{\mathbb{D}}$ and define the operator $Tf(x) = P_\Omega\left(\sum_j f(x_j)\phi_j(x)\right)$, where P_Ω is the orthogonal projection from $L^2(\mathbb{D}, dv)$ onto $\mathbb{PW}_\Omega(\mathbb{D})$. By decreasing r (and thus choosing x_j closer) one can obtain the inequality $\|I - T\| < 1$, in which case T can be inverted by

$$T^{-1}f = \sum_{k=0}^{\infty}(I - T)^k f.$$

For given samples we can calculate Tf and the Neumann series provides the recursion formula $f_{n+1} = f_n + Tf - Tf_n$ and then $\lim_{n\to\infty} f_n = f$ with norm convergence. The rate of convergence is determined by the estimate $\|f_n - f\| \leq \|I - T\|^{n+1}\|f\|$.

The paper [FP05] further provides a necessary condition for the set $\{x_i\}$ to be a sampling set. They find that there is a constant C which is determined by the geometry of $\mathbb{D}$, such that if $r < C^{-1}(1 + |\Omega|^2)^{k/2})^{-1}$ for any $k > 1$, then any (N, r)-lattice $\{x_i\}$ is a sampling set. The paper [CÓ13] obtains similar results, but removes some restrictions on the functions ϕ_j. In particular the partitions of unity do not need to be smooth and can actually be chosen as characteristic functions $\phi_j = \chi_{U_j}$ for a cover of disjoint sets U_j contained in the balls $B(x_j, r/2)$. This is done by lifting the functions to the group of isometries (which in this case is $\mathrm{SU}(1,1)$), and by estimating local oscillations using Sobolev norms for left-invariant vector fields on this group.

6.2.2 Beurling density for Bergman spaces

In this section we describe a collection of celebrated sampling theorems for Bergman spaces on the unit disc by [Sei93] and [Sch97]. Let $\mathcal{H}(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. Let $1 \leq p < \infty$ be given, and equip the unit disc $\mathbb{D}$ with normalized area measure $d\sigma(z)$. We define the Bergman space $A^p(\mathbb{D}) = L^p(\mathbb{D}, d\sigma) \cap \mathcal{H}(\mathbb{D})$. This is a reproducing kernel Banach space with reproducing kernel $K(z, w) = \frac{1}{(1-\overline{w}z)^2}$. By [Sei93] and [Sch97] sampling and interpolation sets for $A^p(\mathbb{D})$ are characterized by the upper and lower Beurling densities

$$D^+(Z) = \limsup_{r\to 1}\sup_{w\in\mathbb{D}} D(\phi_w(Z), r)\,,\ D^-(Z) = \liminf_{r\to 1}\inf_{w\in\mathbb{D}} D(\phi_w(Z), r)\,.$$

Here $\phi_w(z) = \frac{w-z}{1-\overline{w}z}$ and $D(Z, r) = (\sum_{|z_k|<r} \log|z_k|)/(\log(1 - r))$. Let $\rho(z, w) = |\phi_w(z)|$ be the pseudo-hyperbolic distance from z to w, then a sequence $Z = \{z_i\}$ is called uniformly discrete if there is a $\delta > 0$ such that $\rho(z_i, z_j) > \delta$ for $i \neq j$.

Theorem 9 *Let Λ be a set of distinct points in $\mathbb{D}$.*

1.) *A sequence Λ is a set of sampling for A^p if and only if it is a finite union of uniformly discrete sets and it contains a uniformly discrete subsequence Λ' for which $D^-(\Lambda') > 1/p$.*

2.) *A sequence Λ is a set of interpolation for A^p if and only if it is uniformly discrete and $D^+(\Lambda) < 1/p$.*

This results show there can be no Nyquist density for the Bergman spaces, since the sampling sets are always sharply separated from the interpolating sets. We note that the results of Seip and Schuster are for a particular class of holomorphic functions, to which the band-limited functions $\mathbb{PW}_{\Omega(\mathbb{D})}$ do not belong. It is an open question whether it is possible to establish a Nyquist density for band-limited functions on $\mathbb{D}$ and to use this information to create regular lattices and dual lattices determined by the size of the band-limit Ω.

6.2.3 Voronoi Cells and Beurling-Landau Density for $\widehat{\mathbb{D}}$

We develop our model for hyperbolic space on the Poincaré disk $\mathbb{D}$. The motions that preserve lengths in hyperbolic geometry are Möbius-Blaschke maps. Geodesics are subarcs of paths that intersect $\partial\mathbb{D}$ at right angles. Let Γ be a smooth path in the unit disk $\mathbb{D}$. The hyperbolic length of Γ is $\mathcal{L}_H(\Gamma) = \int_\Gamma \frac{2\,|dz|}{1-|z|^2}$. Let $\alpha \in \mathbb{D}$, and let $\varphi_{\theta,\alpha} = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}$ (a Möbius-Blaschke transformation of $\mathbb{D}$ onto $\mathbb{D}$). Then $\varphi_{\theta,\alpha}$ preserves the hyperbolic length, i.e, $\mathcal{L}_H(\varphi_{\theta,\alpha}(\Gamma)) = \mathcal{L}_H(\Gamma)$. The hyperbolic distance ρ between two points z_1, z_2 in $\mathbb{D}$ is

$$\rho(z_1, z_2) = 2\,\operatorname{arctanh}\left(\frac{|z_1 - z_2|}{|1 - \overline{z_2}z_1|}\right) = \log\left(\frac{1 + \frac{|z_1-z_2|}{|1-\overline{z_2}z_1|}}{1 - \frac{|z_1-z_2|}{|1-\overline{z_2}z_1|}}\right) .$$

The distance ρ will be used to determine distance for the sampling lattice Λ. Note that, because we cannot establish the Beurling-Landau densities, we can not create regular lattices and dual lattices.

The Helgason-Fourier transform maps $L^2(\mathbb{D})$ to

$$L^2(\mathbb{R}^+ \times \mathbb{T}\,,\ \frac{1}{2\pi}\lambda\tanh(\lambda\pi/2)d\lambda\,db)\,,$$

which is isomorphic to the space of $L^2(\mathbb{T})$-vector valued square integrable functions with measure $\lambda\tanh(\lambda\pi/2)d\lambda$, in short denoted by

$$L^2(\mathbb{R}^+; L^2(\mathbb{T}), \lambda\tanh(\lambda\pi/2)d\lambda)\,.$$

The negative Laplacian $-\Delta$ is positive with spectrum $\mathbb{R}^+$, and therefore we define Voronoi cells based on a distance on $\mathbb{R}^+$. This distance is denoted dist, and it is an open question in which manner it is related to the measure $\lambda\tanh(\lambda\pi/2)\,d\lambda$. With an appropriate distance function dist, we can define the following.

Definition 8 (Voronoi Cells in $\widehat{\mathbb{D}}$) *Let $\widehat{\Lambda} = \{\widehat{\lambda}_k \in \widehat{\mathbb{D}} = \mathbb{R}^+ \times \mathbb{T} : k \in \mathbb{N}\}$, be a discrete set in frequency space. Then, the Voronoi cells $\{\Phi_k\}$, the Voronoi partition $\mathcal{VP}(\widehat{\Lambda})$, and partition norm $\|\mathcal{VP}(\widehat{\Lambda})\|$ corresponding to this set are defined as follows.*

1.) *The* Voronoi cells $\Phi_k = \{\omega \in \widehat{\mathbb{D}} : \operatorname{dist}(\omega, \widehat{\lambda}_k) \leq \inf_{j\neq k} \operatorname{dist}(\omega, \widehat{\lambda}_j)\}$,

2.) *The* Voronoi partition $\mathcal{VP}(\widehat{\Lambda}) = \{\Phi_k \subseteq \widehat{\mathbb{D}}\}_{k\in\mathbb{Z}^d}$,

3.) *The* partition norm $\|\mathcal{VP}(\widehat{\Lambda})\| = \sup_{k\in\mathbb{Z}^d}\sup_{\omega,\nu\in\Phi_k} \operatorname{dist}(\omega, \nu)$.

A crucial step in answering the question of Nyquist density using Voronoi cells is to determine an appropriate candidate for the distance on $\widehat{\mathbb{D}}$.

6.3 Sampling on the Sphere

One perspective of Fourier analysis is to think of it as a systematic use of symmetry to simplify and understand linear operators. The unit sphere $\mathbb{S}^2$ admits the special orthogonal group of three variables, $SO(3)$ – proper rotations of $\mathbb{R}^3$ about the origin – as a transitive group of symmetries. Fourier analysis on $\mathbb{S}^2$ amounts to the decomposition of $L^2(\mathbb{S}^2)$ into minimal subspaces invariant under all rotations in $SO(3)$. The rotations of the sphere induce operators on functions by rotating the graphs over $\mathbb{S}^2$. The Hilbert space $L^2(\mathbb{S}^2)$ is defined with the usual inner product, using the rotation-invariant area element μ. Background for this section can be found in Driscoll and Healy [DH94], Keiner, Kunis, and Potts [KKP07], and McEwen and Wiaux [MW11].

Band-limited functions on the sphere are spherical polynomials. The corresponding sampling problem is the computation of Fourier coefficients of a function from sampled values at scattered nodes. If we consider the problem of reconstructing a spherical polynomial of degree $N \in \mathbb{N}$ from sample values, one might set up a linear system of equations with $M = (N+1)^2$ interpolation constraints which has to be solved for the unknown vector of Fourier coefficients $\widehat{\mathbf{f}} \in \mathbb{C}^{(N+1)^2}$. If the nodes for interpolation are chosen such that the interpolation problem always has a unique solution, the sampling set is called a fundamental system.

Let $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$ be the two-dimensional unit sphere embedded in $\mathbb{R}^3$. A point $\rho \in \mathbb{S}^2$ is identified in spherical coordinates by $\eta = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))^T$, where the angles (θ, ϕ) are the co-latitude and longitude of η. Topologically, $\mathbb{S}^2 = \widetilde{\mathbb{C}}$. Geodesics are great circles, and the geodesic distance can be most directly written as $\operatorname{dist}(\eta, \xi) = \arccos(\eta \cdot \xi)$. For $\eta, \xi \in \mathbb{S}^2$, $\|\eta - \xi\|_2^2 = 2 - 2(\eta \cdot \xi)$. The distance to the "north pole" $n = (0,0,1)^T$ of $\mathbb{S}^2$ is $\arccos(\eta \cdot n) = \theta$.

The *spherical harmonics* Y_k^n form an o.n. basis for $L^2(\mathbb{S}^2)$ We can define them as follows. The *Legendre polynomials* $P_k : [-1,1] \longrightarrow \mathbb{R}$ are generated by applying the Gram-Schmidt method to $\{x^k\}_{k=0}^{\infty}$. They are given by the Rodrigues formula $P_k(t) = 1/(2^k k!) d^k/dt^k (t^2-1)^k$. The *associated Legendre functions* are defined by

$$P_k^n(t) = \sqrt{\frac{(k-n)!}{(k+n)!}} (t^2-1)^{\frac{n}{2}} \frac{d^n}{dt^n} P_k(t)\,.$$

The *spherical harmonics* $Y_k^n : \mathbb{S}^2 \longrightarrow \mathbb{C}$ of degree $k \in \mathbb{N} \cup \{0\}$ and order $n \in \mathbb{Z}$, $|n| \le k$, are the functions $Y_k^n(\eta) = Y_k^n(\theta, \phi) = \sqrt{\frac{2k+1}{4\pi}} P_k^{|n|}(\cos(\theta)) e^{in\phi}$. We have that

$$\int_0^{2\pi} \int_0^{\pi} Y_k^n(\theta,\phi) Y_l^m(\theta,\phi) \sin(\theta)\, d\theta\, d\phi = \delta_{k,l} \cdot \delta_{m,n}\,,$$

i.e., Y_k^n form an o.n. basis for $L^2(\mathbb{S}^2)$. We say that f is a *spherical polynomial of degree* N if $f(\theta, \phi) = \sum_{k=0}^{N} \sum_{n=-k}^{k} \widehat{f}_k^n Y_k^n$. The space of spherical polynomials of degree at most N has dimension $(N+1)^2$.

The Fourier transform is the spherical Fourier matrix $f(\theta, \phi) = \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \widehat{f}_k^n Y_k^n$, with coefficients given by $\widehat{f}_k^n = \int_{\mathbb{S}^2} f \overline{Y_k^n}\, d\mu$. The dual space of $L^2(\mathbb{S}^2)$ is discrete. The inverse Fourier transform is the construction of a spherical polynomial from the coefficients. The function f is N band-limited ($N \in \mathbb{N}$) if $\widehat{f}_k^n = 0$ for $k > N$. Thus, $f(\theta, \phi) = \sum_{k=0}^{N} \sum_{n=-k}^{k} \widehat{f}_k^n Y_k^n$. For the problem of solving for a spherical polynomial f of degree N from sample values, we are looking to solve for the unknown Fourier coefficients $\{\widehat{f}_k^n\} = \widehat{\mathbf{f}} \in \mathbb{C}^{(N+1)^2}$.

Let $\Lambda = \{\lambda_k\}_{k=1}^{M}$ be a sampling set on $\mathbb{S}^2$. The *mesh norm* δ_Λ and the *separation distance* q_Λ are defined by $\delta_\Lambda = 2 \max_{\eta \in \mathbb{S}^2} \min_{k=1,\dots,M} \operatorname{dist}(\eta, \lambda_k)$, $q_\Lambda = \min_{j \ne k} \operatorname{dist}(\lambda_j, \lambda_k)$. A sampling set Λ is called δ *dense* if for some $0 < \delta \le 2\pi$, $\delta_\Lambda \le \delta$, and called q *separated* if there exists $0 < q \le 2\pi$ such that $q_\Lambda \ge q$. We assume that our sampling set is separated. Finally, a sampling set is called quasi-uniform if there exist a constant C independent of the number on sample points M such that $\delta_\Lambda \le C\, q_\Lambda$.

Sampling on the sphere is how to sample a band-limited function, an Nth degree spherical polynomial, at a finite number of locations, such that all of the information content of the continuous function is captured. Since the frequency domain of a function on the sphere is discrete, the spherical harmonic coefficients describe the continuous function exactly. A sampling theorem thus describes how to exactly recover the spherical harmonic coefficients of the continuous function from its samples. Given Λ, the *spherical Fourier transform matrix* is $\mathbf{Y} = (Y_k^n(\lambda_j))_{j=1,\ldots,M;k=0,\ldots,N;|n|\leq k}$. Let $\mathbf{Y^H}$ denote its complex conjugate transpose. The inverse Fourier transform matrix is the construction of a spherical polynomial of degree N from given data points $(\lambda_j, y_j) \in \mathbb{S}^2 \times \mathbb{C}$ such that the identity $f(\lambda_j) = y_j$ is solved. This is solving the linear system $\mathbf{Y}\widehat{\mathbf{f}} = \mathbf{y}$, $\mathbf{y} = (y_1, y_2, \ldots, y_M)$ for the vector of Fourier coefficients $\widehat{\mathbf{f}} = \{\widehat{f}_k^n\}$ of the spherical polynomial. Essentially, it is the inverse problem to $f = \mathbf{Y}\widehat{\mathbf{f}}$, which corresponds to evaluating a spherical polynomial on Λ.

The open question again is the establishment of the optimal Beurling-Landau densities. This leads to questions about sphere tiling. The papers [KKP07] and [MW11] address the problem of finding optimal sampling lattices.

6.4 General Analytic Surfaces

A surface is a generalization of Euclidean space. From the viewpoint of harmonic analysis, there is a natural interest in both the theory and applications of the study of integrable and square integrable functions on surfaces. Background material for this subsection can be found in Ahlfors [Ahl78, Ahl10], Farkas and Kra [FK80], Forster [For81], Lee [Lee97], and Singer and Thorpe [ST67].

We assume our surfaces are connected and orientable. Therefore, we can choose a coordinate system so that differential forms are positive [ST67]. We consider *Riemann surfaces*, but our discussion carries through to connected and orientable Riemannian manifolds of dimension two [Lee97]. Riemann surfaces allow us to discuss the *Uniformization Theorem*, which gives that all orientable surfaces inherit their intrinsic geometry from their *universal coverings*. There are only three universal covers – the plane $\mathbb{C}$ (Euclidean geometry), the Riemann sphere $\widetilde{\mathbb{C}}$ (spherical geometry), and the hyperbolic disk $\mathbb{D}$ (hyperbolic geometry).

Given connected Riemann surface $\mathcal{S}$ and its universal covering space $\widetilde{\mathcal{S}}$, $\mathcal{S}$ is isomorphic to $\widetilde{\mathcal{S}}/G$, where the group G is isomorphic to the fundamental group of $\mathcal{S}$, $\pi_1(\mathcal{S})$ (see [For81], Section 5). The corresponding universal covering is simply the quotient map which sends every point of $\widetilde{\mathcal{S}}$ to its orbit under G. Thus, the fundamental group of $\mathcal{S}$ determines its universal cover. Moreover, the universal covering is indeed the "biggest" smooth unlimited covering of a connected Riemann surface, in the sense that all other unramified unlimited covering space of a Riemann surface can be covered unlimitedly and without ramification by the universal covering of this surface.

The Uniformization Theorem allows us to classify all universal covers of all Riemann surfaces. This in turn allows us to understand the geometry of every Riemann surface.

Theorem 10 (The Uniformization Theorem) *Let $\mathcal{S}$ be a Riemann surface.*

1.) *Every surface admits a Riemannian metric of constant Gaussian curvature κ.*

2.) *Every simply connected Riemann surface is conformally equivalent to one of the following:*

a.) $\mathbb{C}$ – *Euclidean Geometry* – $\kappa = 0$ – *with isometries* $\left\langle \left\{ e^{i\theta} z + \alpha \right\}, \circ \right\rangle, \theta \in [0, 2\pi)$,

b.) $\widetilde{\mathbb{C}}$ – *Spherical Geometry* – $\kappa = 1$ – *with isometries* $\left\langle \left\{ \frac{\alpha z + \beta}{-\overline{\beta} z + \overline{\alpha}} \right\}, \circ \right\rangle, |\alpha|^2 + |\beta|^2 = 1$,

c.) $\mathbb{D}$ – *Hyperbolic Geometry* – $\kappa = -1$ – *with isometries* $\left\langle \left\{ e^{i\theta} \frac{z-\alpha}{1-\overline{\alpha} z} \right\}, \circ \right\rangle, |\alpha| < 1, \theta \in [0, 2\pi)$.

Given connected Riemann surface $\mathcal{S}$ and its universal covering space $\widetilde{\mathcal{S}}$, $\mathcal{S}$ is isomorphic to $\widetilde{\mathcal{S}}/G$, where the group G is isomorphic to the fundamental group of $\mathcal{S}$, $\pi_1(\mathcal{S})$ (see [For81], Section 5). The corresponding universal covering is simply the quotient map which sends every point of $\widetilde{\mathcal{S}}$ to its orbit under G. Forster [For81] (Section 27) gives the consequences of the Uniformization Theorem very succinctly. The only covering surface of Riemann sphere $\widetilde{\mathbb{C}}$ is itself, with the covering map being the identity. The plane $\mathbb{C}$ is the universal covering space of itself, the once punctured plane $\mathbb{C} \setminus \{z_0\}$ (with covering map $\exp(z - z_0)$), and all tori $\mathbb{C}/\Gamma$, where Γ is a parallelogram generated by $z \longmapsto z + n\gamma_1 + m\gamma_2$, $n, m \in \mathbb{Z}$ and γ_1, γ_2 are two fixed complex numbers linearly independent over $\mathbb{R}$. *The universal covering space of every other Riemann surface is the hyperbolic disk* $\mathbb{D}$. Therefore, the establishment of exact the Beurling-Landau densities for functions in Paley-Wiener spaces in spherical and especially hyperbolic geometries will allow the development of sampling schemes on arbitrary Riemann surfaces.

6.5 The Kunze-Stein phenomenon

In this subsection we will point out that for some amenable subgroups of semi-simple Lie groups, the continuity of convolution operators follows from the famous Kunze-Stein phenomenon [KS60, Cow78, Cow08]. This work is joint with Jens Christensen [CCO15].

Let G be a semi-simple, connected and non-compact group with Haar measure dx, then

Theorem 11 (Cowling) *Suppose that* $1 \leq r \leq \infty$, $1/r + 1/r' = 1$ *and* $f \in L^r(G)$. *Then the mapping* $g \mapsto g * f$ *is bounded from* $L^p(G)$ *to* $L^q(G)$ *provided that one of the following conditions hold:*

1. *if* $r = 1$ *and* $1 \leq p = q \leq \infty$
2. *if* $1 < r \leq 2$,$q \geq r$, $p \leq r'$, $0 \leq 1/p - 1/q \leq 1/r'$ *and* $(p, q) \neq (r, r)$ *and* $(p, q) \neq (r', r')$.
3. *if* $2 < r < \infty$, $q \geq r, p \leq r'$, $0 \leq 1/p - 1/q \leq 1/r'$ *and* $(p, q) \neq (r, r')$
4. *if* $r = \infty$,$p = 1$ *and* $q = \infty$.

We will only focus on the second condition with $p = q$, in which case the condition simplifies to: $L^p * L^r \subseteq L^p$ if $1 < r < 2$ and $r < p < r/(r-1)$.

Since G is semi-simple there is a compact subgroup K, an abelian subgroup A and a nilpotent subgroup N such that $A \times N \times K \ni (a, n, k) \mapsto ank \in G$ is a diffeomorphism and with proper normalizations of Haar measures

$$\int_G f(x)\, dx = \int_K \int_A \int_N f(ank)\, da\, dn\, dk.$$

If f is a K-right invariant function on G, then it can be identified with a function $\widetilde{f}$ on $S = AN$ and

$$\int_G f(x)\, dx = \int_S \widetilde{f}(s)\, ds = \int_A \int_N f(an)\, da\, dn.$$

If f is K-bi-invariant and g is K-right-invariant, then the convolution of the functions $\widetilde{g}$ and $\widetilde{f}$ on $S = AN$ can be written as a convolution of the corresponding functions on G: $\widetilde{g} * \widetilde{f} = g * f$.

From the Kunze-Stein phenomenon we can now obtain continuity of convolution operators on the amenable (subgroup) $S = AN$. This is the second contribution of this paper:

Theorem 12 *Assume that $\widetilde{f}$ is a function on $S = AN$ and that there is a K-bi-invariant function h on G for which $|\widetilde{f}(s)| \leq h(s)$ for all $s \in S$. If h is in $L^r(G)$ for some $r \in (1, 2)$ then $\widetilde{g} \mapsto \widetilde{g} * \widetilde{f}$ is continuous on $L^p(S)$ if $r < p < r/(r-1)$.*

Example 1 *This example shows that when $G = \mathrm{SU}(n, 1)$ convolution by a certain non-$L^1(G)$ function is continuous on $L^p(G)$. This is related to the Bergman projection, and we will follow this path to settling open questions surrounding atomic decompositions and the Bergman projection on unbounded symmetric domains [CR80, BBGR04].*

Any $x \in G$ can be written as a matrix

$$x = \begin{pmatrix} a & b \\ c^t & d \end{pmatrix}$$

where a is an $n \times n$-matrix, $b, c \in \mathbb{C}^n$ and $d \in \mathbb{C}$.

The (multivalued) function

$$f(x) = 1/d^\sigma,$$

*corresponds to a single-valued function $\widetilde{f}$ when restricted to the simply connected subgroup S. If we define $h(x) = |f(x)|$, then h is K-bi-invariant and $|\widetilde{f}(s)| \leq h(s)$. Also, the function h is in $L^r(G)$ for $r > 2n/\sigma$, and therefore convolution by $\widetilde{f}$ is continuous on $L^p(S)$ when $2n/\sigma < p < 2n/(2n - \sigma)$ and $n < \sigma \leq 2n$. Note, that if $n = 1$ this is exactly the statement from Example **??**, since $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SU}(1, 1)$ are isomorphic. Also, if $\sigma > 2n$, then h is in $L^1(G)$ and the continuity of the convolution operator is immediate. One interesting aspect of deriving these convolution continuities on $\mathrm{SU}(n, 1)$ is their relation to the Bergman projection on the unit ball in $\mathbb{C}^n$. In particular they can be used (when applied to weighted $L^p(S)$-spaces) to prove the following classical theorem [Zhu05, Thm. 2.10]*

Theorem 13 *Fix two real parameters a and b and define the integral operator $\mathbf{S}$ by*

$$\mathbf{S}f(z) = (1 - |z|^2)^a \int_{\mathbb{B}^n} \frac{(1 - |w|^2)^b}{|1 - \langle z, w\rangle|^{n+1+a+b}} f(w)\, dv(w),$$

where dv is the volume measure on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ and $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$. Then, for t real and $1 \leq p < \infty$, $\mathbf{S}$ is bounded on $L^p_t(\mathbb{B}^n)$ if and only if $-pa < t + 1 < p(b + 1)$.

We are currently working out the details of this argument and will publish it in a forthcoming paper.

7 Analysis of Point Processes

The analysis of periodic processes is an important area of signal analysis. A probabilistic view of the Riemann zeta function, algorithms on generalizations of Euclidean domains, and variations on equidistribution theory have led to algorithms for several classes of problems in periodic parameter estimation. These algorithms are general and very efficient, and can be applied to the analysis of periodic processes with a single generator and the deinterleaving and analysis of processes with multiple generators.

We divide our analysis into two cases periodic processes created by a single source, and those processes created by several sources. We wish to extract the fundamental period of the generators, and, in the second case, to deinterleave the processes. We first present very efficient algorithm for extracting the fundamental period from a set of sparse and noisy observations of a single source periodic process. The procedure is computationally straightforward, stable with respect to noise and converges quickly. Its use is justified by a theorem which shows that for a set of randomly chosen positive integers, the probability

that they do not all share a common prime factor approaches one quickly as the cardinality of the set increases. The proof of this theorem rests on a probabilistic interpretation of the Riemann zeta function. We then build upon this procedure to deinterleave and then analyze data from multiple periodic processes. This relies both on the probabilistic interpretation of the Riemann zeta function, the equidistribution theorem of Weyl, and Wieners periodogram.

7.1 Single Generator: The MEA Algorithm

We have developed a collection of algorithms that analyze periodic phenomena generated by a single generator. These work even when the data is extremely sparse and noisy. The algorithms use number theory in novel ways to extract the underlying period by modifying the Euclidean algorithm to determine the period from a sparse set of noisy measurements [CS13, SC00, SC98, SC96, Cas97, CS96, CS96]. The elements of the set are the noisy occurrence times of a periodic event with (perhaps very many) missing measurements. The proposed algorithms are computationally straightforward and converge quickly. A robust version is developed that is stable despite the presence of arbitrary outliers. The Euclidean algorithm approach is justified by a theorem which shows that, for a set of randomly chosen positive integers, the probability that they do not all share a common prime factor approaches one quickly as the cardinality of the set increases. The theorem is in essence a probabilistic interpretation of the Riemann Zeta Function. In the noise-free case this implies convergence with only ten data samples, independent of the percentage of missing measurements. In the case of noisy data simulation results show, for example, good estimation of the period from one hundred data samples with fifty percent of the measurements missing and twenty five percent of the data samples being arbitrary outliers.

We then use these algorithms in the analysis of periodic pulse trains, getting an estimate of the underlying period [CS13, SC00, SC98, SC96, Cas97, CS96, CS96]. This estimate, while not maximum likelihood, is used as initialization in a three-step algorithm that achieves the Cramer-Rao bound for moderate noise levels, as shown by comparing Monte Carlo results with the Cramer-Rao bounds. An approach using multiple independent data records is also developed that overcomes high levels of contamination. The data sets arises in radar pulse repetition interval (PRI) analysis, in bit synchronization in communications, in biomedical applications, and other scenarios. We assume our data is a finite set of real numbers $S = \{s_j\}_{j=1}^n$, with $s_j = k_j\tau + \phi + \eta_j$, where τ (the period) is a fixed positive real number, the k_j's are non-repeating positive integers, ϕ (the phase) is a real random variable uniformly distributed over the interval $[0, \tau)$, and the η_j's are zero-mean independent identically distributed (iid) error terms. We assume that the η_j's have a symmetric probability density function (pdf), and that $|\eta_j| < \frac{\tau}{2}$ for all j. We develop an algorithm for isolating the period of the process from this set, which we shall assume is (perhaps very) sparse. In the noise-free case our basic algorithm, given below, is equivalent to the Euclidean algorithm and converges with very high probability given only $n = 10$ data samples, independent of the number of missing measurements. We assume that the original data set is in descending order, i.e., $s_j \geq s_{j+1}$. Let $\widehat{\tau}$ denote the value the algorithm gives for τ, and let "$\longleftarrow$" denote *replacement*, e.g., "$a \longleftarrow b$" means that the value of the variable a is to be replaced by the current value of the variable b.

The Modified Euclidean Algorithm (MEA)

Initialize: Set $\texttt{iter} = 0$.
1.) [Adjoin 0 after first iteration.] If $\texttt{iter} > 0$, then $S \longleftarrow S \cup \{0\}$.
2.) [Form the new set with elements $(s_j - s_{j+1})$.] Set $s_j \longleftarrow (s_j - s_{j+1})$.
3.) Sort the elements in descending order.
4.) [Eliminate noise.] If $0 \leq s_j \leq \eta_0$, then $S \longleftarrow S \setminus \{s_j\}$.
5.) The algorithm terminates if S has only one element s_1. Declare $\widehat{\tau} = s_1$. If not, then set $\texttt{iter} \longleftarrow (\texttt{iter} + 1)$. Go to **(1.)**.

Here, $0 < \eta_0 < \tau$ is a noise threshold. Noise-free simulation examples demonstrate successful estimation of τ for $n = 10$ with 99.99% of the possible measurements missing. In fact, with only 10 data samples, it is possible to have the percentage of missing measurements arbitrarily close to 100%. There is, of course, a cost, in that the number of iterations the algorithm needs to converge increases with the percentage of missing measurements. In the presence of noise and false data (outliers), there is a tradeoff between the number of data samples, the amount of noise, and the percentage of outliers. The algorithm will perform well given low noise for $n = 10$, but will degrade as noise is increased. However, given more data, it is possible to reduce noise effects and speed up convergence by binning the data, and averaging across bins. Binning can be effectively implemented by using an adaptive threshhold with a gradient operator, allowing convergence in a single iteration in many cases. Simulation results show, for example, good estimation of the period from one hundred data samples with fifty percent of the measurements missing and twenty five percent of the data samples being arbitrary outliers [CS13, CS96].

Our algorithm is based on several theoretical results, which we now present. First, we can modify the basic Euclidean algorithm, allowing a reformulation using subtraction rather than division, because $\gcd(\tau k_1, \ldots, \tau k_n) = \tau \gcd((k_1 - k_2), \ldots, (k_{n-1} - k_n), k_n)$. We then show that our procedure almost surely converges to the period by proving the following result. The Riemann Zeta Function is defined on the complex half space $\{z \in \mathbb{C} : \Re(z) > 1\}$ by $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$. Euler demonstrated the connection of ζ with number theory by showing that $\zeta(z) = \prod_{j=1}^{\infty} \frac{1}{1-(p_j)^{-z}}$, $\Re(z) > 1$, where $\mathbb{P} = \{p_1, p_2, p_3, \ldots\} = \{2, 3, 5, \ldots\}$ is the set of all prime numbers. In the following, we let $P\{\cdot\}$ denote probability, $\mathrm{card}\{\cdot\}$ denote the cardinality of the set $\{\cdot\}$, and let $\{1, \ldots, \ell\}^n$ denote the sublattice of positive integers in $\mathbb{R}^n$ with coordinates c such that $1 \le c \le \ell$. Therefore, $N_n(\ell) = \mathrm{card}\{(k_1, \ldots, k_n) \in \{1, \ldots, \ell\}^n : \gcd(k_1, \ldots, k_n) = 1\}$ is the number of relatively prime elements in $\{1, \ldots, \ell\}^n$.

Theorem 14 ([CS13, CS96]) *For $n \ge 2$, we have that $\lim_{\ell \to \infty} \frac{N_n(\ell)}{\ell^n} = [\zeta(n)]^{-1}$. Therefore, given n ($n \ge 2$) randomly chosen positive integers $\{k_1, \ldots, k_n\}$, $P\{\gcd(k_1, \ldots, k_n) = 1\} = [\zeta(n)]^{-1}$. Also, $\lim_{n \to \infty} [\zeta(n)]^{-1} = 1$, converging to 1 from below faster than $(1 - 2^{1-n})$.*

Thus, as n grows it quickly becomes very likely that n randomly selected integers have a gcd of 1. This fact, together with Proposition 1, make estimation of τ via our algorithm possible.

The parameter estimation techniques given above lead to an effective method for periodic pulse interval analysis (see [CS13, SC00, SC98, SC96, Cas97, CS96, CS96]). We assume time is highly resolved and ignore any time quantization error. We are primarily concerned with a single periodic pulse train with (perhaps very many) missing observations that may be contaminated with outliers. Our data model for this case, in terms of the arrival times t_j, is given as above, with the additional assumption that η_j is zero-mean additive white Gaussian noise. Outliers are included as arbitrary arrival times. The problem, again, is to recover the period τ and possibly the phase ϕ. With Gaussian noise the minimum variance unbiased estimates for this linear regression problem take a least-squares form. However, this requires knowledge of the k_j's. We therefore have developed a multi-step procedure that proceeds by (i) estimating τ directly, (ii) estimating the k_j's, and (iii) refining the estimate of τ using the estimated k_j's in the least-squares solution.

7.2 Multiple Generators: The EQUIMEA Algorithm

We close by discussing our work on deinterleaving. Our data model is the union of M copies of our previous datasets, each with different periods or "generators" $\Gamma = \{\tau_i\}$, k_{ij}'s and phases. Let $\tau = \max_i\{\tau_i\}$. Then our data is $S = \bigcup_{i=1}^{M} \{\phi_i + k_{ij}\tau_i + \eta_{ij}\}_{j=1}^{n_i}$, where n_i is the number of elements from the i^{th} generator, $\{k_{ij}\}$ is a linearly increasing sequence of natural numbers with missing observations, ϕ_i is a random variable uniformly distributed in $[0, \tau_i)$, and the η_{ij}'s are zero-mean iid Gaussian with standard deviation

$3\sigma_{ij} < \tau/2$. We think of the data as events from M periodic processes, and represent it, after reindexing, as $S = \{\alpha_l\}_{l=1}^{N}$. We difference as in the MEA, but we compute **all** of the differences. We repeat this m times, and saving the elements from each iteration. We form a union of all of these data elements. The relative primeness of data generated by one generator will "fill in" the missing elements for that generator, whereas the data from two different generators will become "Weyl flat." The number of times m that this process is applied will be determined experimentally. Assuming only minimal knowledge of the range of $\{\tau_i\}$, namely bounds T_L, T_U such that $0 < T_L \leq \tau_i \leq T_U$, we phase wrap the data by the mapping $\Phi_\rho(\alpha_l) = \left\langle \frac{\alpha_l}{\rho} \right\rangle = \frac{\alpha_l}{\rho} - \left\lfloor \frac{\alpha_l}{\rho} \right\rfloor$, where $\rho \in [T_L, T_U]$, and $\lfloor \cdot \rfloor$ is the floor function. Thus $\langle \cdot \rangle$ is the fractional part, and so $\Phi_\rho(\alpha_l) \in [0, 1)$.

Definition 9 *A sequence of real random variables* $\{x_j\} \subset [0, 1)$ *is* essentially uniformly distributed in the sense of Weyl *if given* $a, b,\ 0 \leq a < b < 1$, $\frac{1}{n}\text{card}\{1 \leq j \leq n : x_j \in [a, b]\} \longrightarrow (b - a)$ *as* $n \longrightarrow \infty$ *almost surely.*

Weyl's Theorem is presented in [DM72]. For our variation, we assume that for each i, $\{k_{ij}\}$ is a linearly increasing infinite sequence of natural numbers with missing observations such that $k_{ij} \longrightarrow \infty$ as $j \longrightarrow \infty$. We must make this assumption because the result is only approximately true for a finite length sequence.

Theorem 15 *For almost every choice of* ρ *(in the sense of Lebesgue measure)* $\Phi_\rho(\alpha_l)$ *is essentially uniformly distributed in the sense of Weyl.*

Moreover, the set of ρ's for which this is not true are rational multiples of $\{\tau_i\}$. Therefore, except for those values, $\Phi_\rho(\alpha_{ij})$ is essentially uniformly distributed in $[0, 1)$. Moreover, the values at which $\Phi_\rho(\alpha_{ij}) = 0$ almost surely are $\rho \in \{\tau_i/n : n \in \mathbb{N}\}$. These values of ρ cluster at zero, but spread out for lower values of n. We phase wrap the data by computing modulus of the spectrum, i.e., compute $|S_{iter}(\tau)| = |\sum_{j=1}^{N} e^{(2\pi i s(j)/\tau)}|$. The values of $|S_{iter}(\tau)|$ will have peaks at the periods τ_j and their harmonics $(\tau_j)/k$. The "noise-like" behavior of $\Phi_\rho(\alpha_l)$ for *a.e.* ρ leads to a "flat" range for S for $\rho \notin \{\tau_i/n : n \in \mathbb{N}\}$. In turn, this gives the following. Let i_0 denote the index of the most prolific generator. We then isolate the data generated by τ_{i_0} by convolution with a pulse train of width τ_{i_0}, and subtract it out of Δ_0. We then repeat the process, terminating when Δ_0 equals the empty set. We refer to this at the EQIMEA algorithm.

The EQUIMEA Algorithm – Multiple Periods

Initialize: Sort the elements of S in descending order. Form the new set with elements $(s_l - s_{l+1})$. Set $s_l \longleftarrow (s_l - s_{l+1})$. (Note, this eliminates the phase φ.) Set $\texttt{iter} = 1$, $i = 1$, and $\texttt{Error}$. Go to **1.)**
1.) [Adjoin 0 after first iteration.] $S_{iter} \longleftarrow S \cup \{0\}$.
2.)[Sort.] Sort the elements of S_{iter} in descending order.
3.)[Compute all differences.] Set $S_{iter} = \bigcup(s_j - s_k)$ with $s_j > s_k$.
4.) [Eliminate zero(s).] If $s_j = 0$, then $S_{iter} \longleftarrow S_{iter} \setminus \{s_j\}$.
5.)[Adjoin previous iteration.] Form $S_{iter} \longleftarrow S_{iter} \cup S_{iter-1}$.
6.)][Compute spectrum.] Compute $|S_{iter}(\tau)| = |\sum_{j=1}^{N} e^{(2\pi i s(j)/\tau)}|$.
7.)[Threshold.] Choose the largest peak. Label it as τ_{iter}
8.) If $|\tau_{iter} - \tau_{iter-1}| <$ Error. Declare $\widehat{\tau_i} = \tau_{iter}$. If not, $\texttt{iter} \longleftarrow (\texttt{iter} + 1)$. Go to **1.)**.
9.) Given τ_i, frequency notch $|S_{iter}(\tau)|$ for $\widehat{\tau_i}/m$, $m \in \mathbb{N}$. Let $i \longleftarrow i + 1$.
10.)[Compute spectrum.] Compute $|S_{iter}(\tau)| = |\sum_{j=1}^{N} e^{(2\pi i s(j)/\tau)}|$.
11.)[Threshold.] Choose the largest peak. Label it as τ_{i+1}. Algorithm terminates when there are no peaks. Else, go to **9.)**.

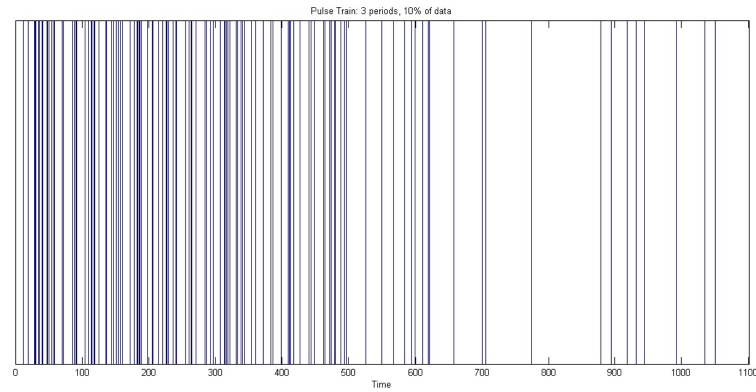

Figure 1: Three Periods – OriginalData

We now demonstrate the algorithm. The original data had three underlying periods – $1, (1+\sqrt{5})/2, \sqrt{7}$, with 90% of the information randomly removed, and 10% jitter noise.
Here is the $|S_{iter}|$ after two iterations.

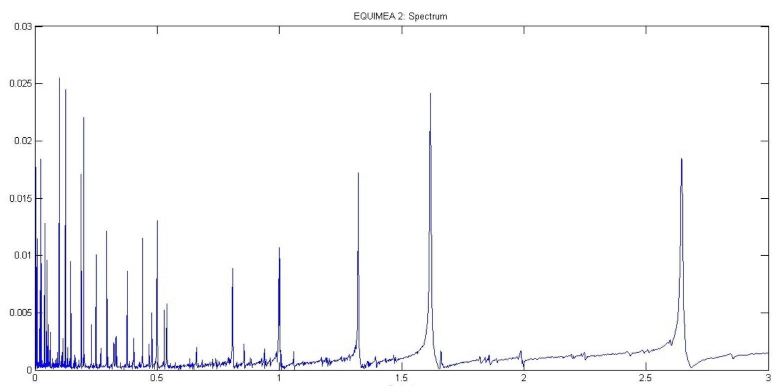

Figure 2: EQUIMEA – Three Periods – Iter2 – Spectrum

There are several research items that need to be addressed. First, we need to explore applications of both algorithms to general point processes. The second is to combine this approach with a localized time-frequency to explore signal separation algorithms. We plan on looking at waveform signatures in, e.g., EEG data (using wavelets), and see the application of the combination of time-frequency analysis with these point process algorithms to the deinterleaving and analysis.

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